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General boundary conditions for the $sl(\mathcal{N})$ and $sl(\mathcal{M}|\mathcal{N})$ open spin chains

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Abstract

Two types of boundary conditions ('soliton preserving' and 'soliton non-preserving') are investigated for the $sl(\mathcal{N})$ and $sl(\mathcal{M}|\mathcal{N})$ open spin chains. The appropriate reflection equations are formulated and the corresponding solutions are classified. The symmetry and the Bethe Ansatz equations are derived for each case.

The general treatment for non-diagonal reflection matrices associated to 'soliton preserving' case is worked out. The connection between the 'soliton non-preserving' boundary conditions and the twisted (super) Yangians is also discussed.

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Introduction

The possibility of constructing and (at least partially) solving by algebraic and/or analytical methods, one-dimensional interacting quantum spin chains, is one of the major achievements in the domain of quantum integrable systems. Its main tool is the quantum R-matrix, obeying a cubic Yang-Baxter equation, the "coproduct" properties of which allow the building of an L-site transfer matrix with identical exchange relations and the subsequent derivation of quantum commuting Hamiltonians [1]. The same structure is instrumental in formulating the quantum inverse scattering procedure, initiated by the Leningrad school [2].

A subsequent development was the definition of exactly solvable open spin chains with non-trivial boundary conditions. These are characterised by a second object: the reflection matrix K, obeying a quadratic consistency equation with the R matrix, with the generic abstract form $RKRK = KRKR$ [3–7]. Using again "coproduct-like" properties of this structure one constructs suitable transfer matrices yielding (local) commuting spin chain Hamiltonians by combining K and semi-tensor products of R [4].

Many efforts have been devoted to this issue [8–23], based on the pioneering approach of Sklyanin and we here aim at treating a particular, but very significant, class of examples for this problem.

To better characterise the type of spin chain which we will be considering here it is important to recall that both R and K matrices have an interpretation in terms of diffusion theory for particle-like objects identified in several explicit cases with exact eigenstates of some quantum integrable field theories such as sine–Gordon [24], non-linear Schrödinger equation [4, 25] or principal chiral model [26]. R describes the basic 2-body scattering amplitudes and K describes a 1-body scattering or reflection on a boundary. The Yang-Baxter equation (YBE) and reflection equation (RE) then characterise consistent factorisability of any L-body amplitude in terms of 1- and 2-body scattering amplitudes, regardless of the order of occurrence of the 1- and 2-body events in the diffusion process.

As a consequence, when one describes the scattering theory of a model with more than one type of particle involved, one is led to introduce several operators of R and K type. The case which we examine here corresponds to a situation where the states involved can be split into resp. particles and anti-particles with a suitable representation of CP transformation acting on the states. Within the context of integrable field theories it is justified to denote them respectively "solitons" S and "antisolitons" A. Assuming that the 2-body diffusion conserves the soliton or antisoliton nature of the particles, but that the reflection may change it, one should therefore consider four types of R matrices (resp. R^{SS}_{SS}, R^{SA}_{AS}, R^{AS}_{SA}, R^{AA}_{AA}) connected by CP operations; and four reflection matrices (resp. K^{S}_{S}, K^{S}_{A}, K^{A}_{S}, K^{A}_{A}). It then becomes possible to define several non-equivalent constructions of commuting transfer matrices. As a consequence, one sees that a variety of spin chain models can be built using a Sklyanin-like procedure, depending on which transfer matrix is being constructed and which reflection matrices are used to build it. Locality arguments also come into play, leading to more complicated combinations of transfer matrices as we shall presently see.

We shall here describe the construction, and present the resolution by analytical Bethe Ansatz methods [16, 27], of open spin chains based on the simplest rational R matrix solutions of the Yang-Baxter equations for underlying $sl(\mathcal{N})$ and $sl(\mathcal{M}|\mathcal{N})$ Lie (super) algebras. These solutions, with a rational dependence on the spectral parameter, are instrumental in defining the Yangian [28]. We shall consider two types of associated reflection matrices K to build two distinct types of integrable spin chains: one which entails purely soliton- and antisoliton-preserving reflection amplitude by two matrices K_S^S and K_A^A (hereafter denoted "soliton-preserving case" or SP); the other which entails the two soliton-non-preserving reflection amplitudes K_S^A and K_A^S (hereafter denoted "soliton-non-preserving case" or SNP). Closed spin chains based on $sl(\mathcal{M}|\mathcal{N})$ superalgebras were studied in e.g. [29] and, in the case of alternating fundamental-conjugate representations of $sl(\mathcal{M}|\mathcal{N})$ in [30]. Open spin chains based on $sl(1|2)$ have been studied in details in e.g. [21, 22].

The plan of our presentation is as follows:

We first define the relevant algebraic objects, R matrix and K matrix, together with their compatibility (Yang–Baxter and reflection) equations and their relevant properties. In particular we introduce the various reflection equations which arise in the SP and SNP cases. Let us emphasise that all notions introduced in the $sl(\mathcal{N})$ case will be straightforwardly generalised to the $sl(\mathcal{M}|\mathcal{N})$ case, albeit with a graded tensor product.

In a second part we define the commuting transfer matrices which can be built in both cases, and the local Hamiltonians which can be built from them. Locality requirement leads to considering a product of two transfer matrices, resp. soliton-antisoliton and antisoliton-soliton, in the SNP case. Once again this construction will be valid, with suitable modifications, for the $sl(\mathcal{M}|\mathcal{N})$ case.

In a third part we discuss the symmetries of these transfer matrices induced by their respective YBE and RE structures, in particular focusing on the connection between the SNP case and twisted Yangians.

We then start the discussion of the analytical Bethe Ansatz formulation for the $sl(\mathcal{N})$ spin chains in the SNP case. The derivation of suitable new fusion formulae explicited in Appendix A and B makes it possible to get a set of Bethe equations.

In Section 5 we consider the case of $sl(\mathcal{M}|\mathcal{N})$ super algebra as underlying algebra. Contrary to the previous case it is first needed to establish a classification for the reflection matrices based on the rational (super Yangian) quantum R-matrix solution, both for SP and SNP conditions. We then establish the Bethe equations for both SP and SNP cases. In the SP case in addition we consider spin chains built from general K matrix solutions, in the SNP case we restrict ourselves to diagonal K matrices.

1 Yang–Baxter and reflection equations

The R and K matrices obey sets of coupled consistency equations together with characteristic properties which we now describe.

1.1 The R matrix

We will consider in a first stage the $sl(\mathcal{N})$ invariant R matrices

$$R_{12}(\lambda) = \lambda \mathbb{I} + i\mathcal{P}_{12} \tag{1.1}$$

where $\mathcal{P}$ is the permutation operator

$$\mathcal{P}_{12} = \sum_{i,j=1}^{\mathcal{N}} E_{ij} \otimes E_{ji} \; . \tag{1.2}$$

E_{ij} are the elementary matrices with 1 in position (i,j) and 0 elsewhere.
We define a transposition ^{t} which is related to the usual transposition ^{T} by (A is any matrix):

$$A^t = V^{-1}\, A^T\, V \qquad \text{where} \quad \begin{cases} V = \text{antidiag}(1,1,\dots,1)\,, & \text{for which} \;\; V^2 = \theta_0 = 1 \\ \text{or} & \\ V = \text{antidiag}\Big(\underbrace{1,\dots,1}_{\mathcal{N}/2}, \underbrace{-1,\dots,-1}_{\mathcal{N}/2}\Big)\,, & \text{for which} \;\; V^2 = \theta_0 = -1\,. \end{cases} \tag{1.3}$$

The second case is forbidden for $\mathcal{N}$ odd.
This R matrix satisfies the following properties:

(i) Yang–Baxter equation [1, 31–33]

$$R_{12}(\lambda_1-\lambda_2)\; R_{13}(\lambda_1)\; R_{23}(\lambda_2) = R_{23}(\lambda_2)\; R_{13}(\lambda_1)\; R_{12}(\lambda_1-\lambda_2) \tag{1.4}$$

(ii) Unitarity

$$R_{12}(\lambda)\; R_{21}(-\lambda) = \zeta(\lambda) \tag{1.5}$$

where $R_{21}(\lambda) = \mathcal{P}_{12}R_{12}(\lambda)\mathcal{P}_{12} = R_{12}^{t_1t_2}(\lambda) = R_{12}(\lambda)$.

(iii) Crossing-unitarity

$$R_{12}^{t_1}(\lambda)\; R_{12}^{t_2}(-\lambda-2i\rho) \;\; = \bar{\zeta}(\lambda+i\rho) \tag{1.6}$$

where $\rho = \frac{\mathcal{N}}{2}$ and

$$\zeta(\lambda) = (\lambda+i)(-\lambda+i), \qquad \bar{\zeta}(\lambda) = (\lambda+i\rho)(-\lambda+i\rho). \tag{1.7}$$

It obeys

$$[A_1A_2,\; R_{12}(\lambda)] = 0 \qquad \text{for any matrix } A. \tag{1.8}$$

The R matrix can be interpreted physically as a scattering matrix [24, 33, 34] describing the interaction between two solitons that carry the fundamental representation of $sl(\mathcal{N})$.

To take into account the existence, in the general case, of anti-solitons carrying the conjugate representation of $sl(\mathcal{N})$, we shall introduce another scattering matrix, which describes the interaction between a soliton and an anti-soliton. This matrix is derived as follows

$$\begin{aligned} R_{\bar{1}2}(\lambda) = \bar{R}_{12}(\lambda) \;\; &:= \;\; R_{12}^{t_1}(-\lambda-i\rho) & (1.9) \\ &= \;\; R_{12}^{t_2}(-\lambda-i\rho) \;\; =: \; R_{1\bar{2}}(\lambda) = \bar{R}_{21}(\lambda) & (1.10) \end{aligned}$$

In the case $\mathcal{N}=2$ and for $\theta_0=-1$ ($sp(2)$ case), $\bar{R}$ is proportional to R, so that there is no genuine notion of anti-soliton. This reflects the fact that the fundamental representation of $sp(2)=sl(2)$ is self-conjugate. This does not contradict the fact that for $\mathcal{N}=2$ and for $\theta_0=+1$ ($so(2)$ case), there exists a notion of soliton and anti-soliton.

The equality between $R_{\bar{1}2}(\lambda)$ and $R_{1\bar{2}}(\lambda)$ in (1.9) reflects the CP invariance of R, from which one also has $R_{\bar{1}\bar{2}}=R_{12}$, i.e. the scattering matrix of anti-solitons is equal to the scattering matrix of solitons. In (1.9), $\bar{R}_{12}(\lambda)=(-\lambda-i\rho)\mathbb{I}+iQ_{12}$, in which Q_{12} is proportional to a projector onto a one-dimensional space. It satisfies

$$Q^2=2\rho\, Q \qquad \text{and} \qquad \mathcal{P}\, Q=Q\, \mathcal{P}=\theta_0 Q\ . \tag{1.11}$$

The $\bar{R}$ matrix (1.10) also obeys

(i) A Yang–Baxter equation

$$\bar{R}_{12}(\lambda_1-\lambda_2)\ \bar{R}_{13}(\lambda_1)\ R_{23}(\lambda_2)=R_{23}(\lambda_2)\ \bar{R}_{13}(\lambda_1)\ \bar{R}_{12}(\lambda_1-\lambda_2) \tag{1.12}$$

(ii) Unitarity

$$\bar{R}_{12}(\lambda)\ \bar{R}_{21}(-\lambda)=\bar{\zeta}(\lambda) \tag{1.13}$$

(iii) Crossing-unitarity

$$\bar{R}_{12}^{t_1}(\lambda)\ \bar{R}_{12}^{t_2}(-\lambda-2i\rho)\ =\zeta(\lambda+i\rho)\ . \tag{1.14}$$

Remark: The crossing-unitarity relation written in the literature usually involves a matrix $M=V^TV$. In our case, M turns out to be 1 for two reasons: (i) the factors q^k of the quantum (trigonometric) case degenerate to 1 in the Yangian (rational) case; (ii) the signs usually involved in the super case are in this paper (section 5) taken into account in the definition of the super-transposition (5.3).

1.2 The K matrix

The second basic ingredient to construct the open spin chain is the K matrix. We shall describe in what follows two different types of boundary conditions, called *soliton preserving* (SP) [8–11] and *soliton non–preserving* (SNP) [12–14].

1.2.1 Soliton preserving reflection matrices

In the case of soliton preserving boundary conditions, the matrix K is a numerical solution of the reflection (boundary Yang–Baxter) equation [3]

$$R_{ab}(\lambda_a-\lambda_b)\ K_a(\lambda_a)\ R_{ba}(\lambda_a+\lambda_b)\ K_b(\lambda_b)=K_b(\lambda_b)\ R_{ab}(\lambda_a+\lambda_b)\ K_a(\lambda_a)\ R_{ba}(\lambda_a-\lambda_b)\ , \tag{1.15}$$

and it describes the reflection of a soliton on the boundary, coming back as a soliton.

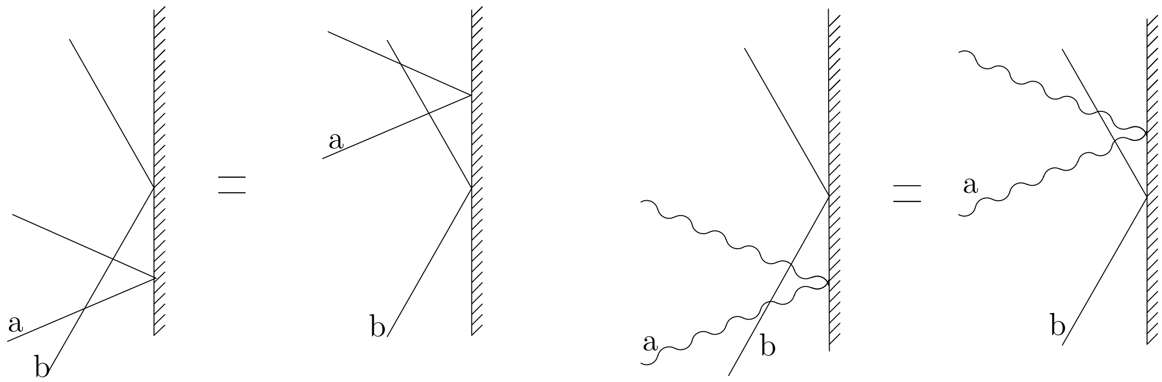
Another reflection equation is required for what follows, in particular for the 'fusion' procedure described in the appendices

$$\bar{R}_{ab}(\lambda_a - \lambda_b)\ K_{\bar{a}}(\lambda_a)\ \bar{R}_{ba}(\lambda_a + \lambda_b)\ K_b(\lambda_b) = K_b(\lambda_b)\ \bar{R}_{ab}(\lambda_a + \lambda_b)\ K_{\bar{a}}(\lambda_a)\ \bar{R}_{ba}(\lambda_a - \lambda_b)\ . \qquad (1.16)$$

$K_{\bar{a}}$ is a solution of the anti-soliton reflection equation obtained from (1.15) by CP conjugation and actually identical to (1.15) due to the CP invariance of the R-matrix. It describes the reflection of an anti-soliton on the boundary, coming back as an anti-soliton.

Equation (1.16) appears as a criterion for a consistent choice of a couple of solutions K_a and $K_{\bar{a}}$ of (1.15), yielding the commutation of transfer matrices (hereafter to be defined).

Graphically, (1.15) and (1.16) are represented as follows:



These K matrices (solutions of the soliton preserving reflection equation (1.15)) have been classified for $sl(\mathcal{N})$ Yangians in [35]. This classification can be recovered as a particular case of our proposition 5.1, where the $sl(\mathcal{M}|\mathcal{N})$ Yangians are studied. Yang–Baxter and reflection equations will indeed take the same form albeit with a graded tensor product in the superalgebraic case (see section 5 for more details). Proposition 5.2 then provides the classification of pairs $\{K_a(\lambda), K_{\bar{a}}(\lambda)\}$ which obey (1.15) and the compatibility equation (1.16).

1.2.2 Soliton non-preserving reflection matrices

In the context of soliton non-preserving boundary conditions one has to consider [12–14] the case where a soliton reflects back as an anti-soliton. The corresponding reflection equation has the form

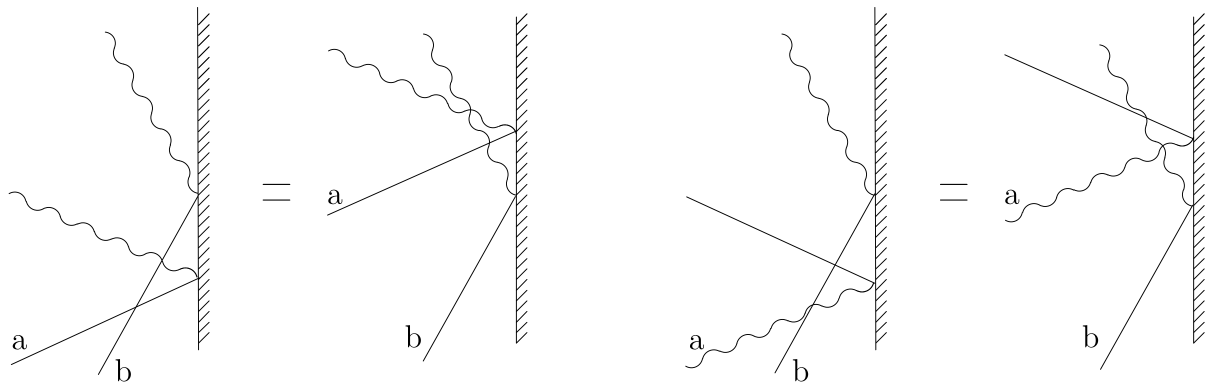
$$R_{ab}(\lambda_a - \lambda_b)\ \widetilde{K}_a(\lambda_a)\ \bar{R}_{ba}(\lambda_a + \lambda_b)\ \widetilde{K}_b(\lambda_b) = \widetilde{K}_b(\lambda_b)\ \bar{R}_{ab}(\lambda_a + \lambda_b)\ \widetilde{K}_a(\lambda_a)\ R_{ba}(\lambda_a - \lambda_b). \qquad (1.17)$$

Note that equation (1.17) is satisfied by the generators of the so-called *twisted Yangian* [36,37], which will be discussed in section 3.2.

Similarly to the previous case, one introduces $\widetilde{K}_{\bar{a}}$, describing an anti-soliton that reflects back as a soliton, satisfying (1.17) and the consistency condition

$$\bar{R}_{ab}(\lambda_a - \lambda_b)\ \widetilde{K}_{\bar{a}}(\lambda_a)\ R_{ba}(\lambda_a + \lambda_b)\ \widetilde{K}_b(\lambda_b) = \widetilde{K}_b(\lambda_b)\ R_{ab}(\lambda_a + \lambda_b)\ \widetilde{K}_{\bar{a}}(\lambda_a)\ \bar{R}_{ba}(\lambda_a - \lambda_b)\ . \qquad (1.18)$$

Graphically, (1.17) and (1.18) are represented as follows:



The K matrices corresponding to the soliton non-preserving reflection equation are classified in proposition 5.3, in the case of $sl(\mathcal{M}|\mathcal{N})$ Yangians, where once again similar Yang–Baxter and reflection equations occur. Proposition 5.4 then provides the classification of pairs $\{\widetilde{K}_a(\lambda), \widetilde{K}_{\bar{a}}(\lambda)\}$ which obey (1.17) and the compatibility equation (1.18).

2 The transfer matrix

We are now in a position to build open spin chains with different boundary conditions from the objects K, $\widetilde{K}$, R and $\bar{R}$ [4]. Our purpose is to determine the spectrum and the symmetries of the transfer matrix for the case where soliton non-preserving boundary conditions are implemented. We first recall the general settings for the soliton preserving case.

2.1 Soliton preserving case

Let us first define the transfer matrix for the well-known boundary conditions, i.e. the soliton preserving ones. The starting point is the construction of the monodromy matrices

$$\mathcal{T}_a(\lambda) = T_a(\lambda)\ K_a^-(\lambda)\ \hat{T}_a(\lambda), \tag{2.1}$$

$$\overline{\mathcal{T}}_a(\lambda) = T_{\bar{a}}(\lambda)\ K_{\bar{a}}^-(\lambda)\ \hat{T}_{\bar{a}}(\lambda) \tag{2.2}$$

and two transfer matrices (soliton–soliton and anti-soliton–anti-soliton)

$$t(\lambda) = \mathrm{Tr}_a\, K_a^+(\lambda)\ \mathcal{T}_a(\lambda), \qquad \bar{t}(\lambda) = \mathrm{Tr}_a\, K_{\bar{a}}^+(\lambda)\ \overline{\mathcal{T}}_a(\lambda)\ , \tag{2.3}$$

with

$$\begin{aligned} T_a(\lambda)&=R_{aL}(\lambda)\dots R_{a1}(\lambda)\ , \qquad & \hat{T}_a(\lambda)&=R_{1a}(\lambda)\dots R_{La}(\lambda)\ , \\ T_{\bar{a}}(\lambda)&=\bar{R}_{aL}(\lambda)\dots \bar{R}_{a1}(\lambda)\ , \qquad & \hat{T}_{\bar{a}}(\lambda)&=\bar{R}_{1a}(\lambda)\dots \bar{R}_{La}(\lambda)\ . \end{aligned} \tag{2.4}$$

The numerical matrices $K_a^-(\lambda)$, $K_{\bar{a}}^-(\lambda)$ are solutions of (1.15), (1.16) and K_a^+ satisfies a reflection equation 'dual' to (1.15),

$$\begin{aligned} &R_{ab}(-\lambda_a+\lambda_b)\ K_a^+(\lambda_a)^t\ R_{ba}(-\lambda_a-\lambda_b-2i\rho)\ K_b^+(\lambda_b)^t \\ &\qquad = K_b^+(\lambda_b)^t\ R_{ab}(-\lambda_a-\lambda_b-2i\rho)\ K_a^+(\lambda_a)^t\ R_{ba}(-\lambda_a+\lambda_b)\ . \end{aligned} \tag{2.5}$$

The solutions of (2.5) take the form $K^+_a(\lambda) = K^t_a(-\lambda - i\rho)$, where $K_a(\lambda)$ is a solution of (1.15). Similarly $K^+_{\bar a}(\lambda)$ satisfies a reflection equation dual to (1.15), the solutions of which being of the form $K^+_{\bar a}(\lambda) = K^t_{\bar a}(-\lambda - i\rho)$. In addition, $K^+_{\bar a}$ satisfies also a compatibility condition dual to (1.16). Actually, the dual reflection equations happen to be the usual reflection equations after a redefinition $\lambda_c \to -\lambda_c - i\rho$.

From their explicit expression, one can deduce that the monodromy matrices $\mathcal{T}(\lambda)$ and $\overline{\mathcal{T}}(\lambda)$ obey the following equations:

$$
\begin{aligned}
R_{ab}(\lambda_a-\lambda_b)\ \mathcal{T}_a(\lambda_a)\ R_{ba}(\lambda_a+\lambda_b)\ \mathcal{T}_b(\lambda_b) &= \mathcal{T}_b(\lambda_b)\ R_{ab}(\lambda_a+\lambda_b)\ \mathcal{T}_a(\lambda_a)\ R_{ba}(\lambda_a-\lambda_b) &(2.6)\\
R_{ab}(\lambda_a-\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ R_{ba}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_b(\lambda_b) &= \overline{\mathcal{T}}_b(\lambda_b)\ R_{ab}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ R_{ba}(\lambda_a-\lambda_b) &(2.7)\\
\bar R_{ab}(\lambda_a-\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ \bar R_{ba}(\lambda_a+\lambda_b)\ \mathcal{T}_b(\lambda_b) &= \mathcal{T}_b(\lambda_b)\ \bar R_{ab}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ \bar R_{ba}(\lambda_a-\lambda_b)\ , &(2.8)
\end{aligned}
$$

which just correspond to the soliton preserving reflection equations (1.15) and (1.16). One can recognise, in the relation (2.6), the exchange relation of reflection algebras based on R-matrix of $\mathcal{Y}(sl_{\mathcal{N}})$. The form of $K^\pm$ determines the precise algebraic structure which is involved, see section 3.

As usual in the framework of spin chain models, the commutativity of the transfer matrices (2.3)

$$[t(\lambda), t(\mu)] = 0 \tag{2.9}$$

$$[\bar t(\lambda), \bar t(\mu)] = 0 \qquad\qquad [t(\lambda), \bar t(\mu)] = 0 \tag{2.10}$$

is ensured by the above exchange relations. The first commutator guarantees the integrability of the model, whose Hamiltonian is given by

$$\mathcal{H} = -\frac{1}{2}\frac{d}{d\lambda} t(\lambda)\Big|_{\lambda=0}\ , \tag{2.11}$$

the locality being ensured because $R(0) = \mathcal{P}$.

The commutators (2.10) will be needed so that the fusion procedure be well-defined (see appendices). The transfer matrix for $K^+(u) = K^-(u) = 1$ satisfies a crossing-like relation (see for details [14, 27])

$$t(\lambda) = \bar t(-\lambda - i\rho)\ . \tag{2.12}$$

The eigenvalues of the transfer matrices as well as the corresponding Bethe Ansatz equations have been derived for diagonal K matrices in [8, 16].

2.2 Soliton non-preserving case

This case was studied in [14] for the $sl(3)$ chain only. Here we generalise the results for any $sl(\mathcal{N})$. One introduces two monodromy matrices

$$
\begin{aligned}
\mathcal{T}_a(\lambda) &= T_a(\lambda)\ \widetilde{K}^-_a(\lambda)\ \hat T_{\bar a}(\lambda)\ ,\\
\overline{\mathcal{T}}_a(\lambda) &= T_{\bar a}(\lambda)\ \widetilde{K}^-_{\bar a}(\lambda)\ \hat T_a(\lambda)\ ,
\end{aligned} \tag{2.13}
$$

and two transfer matrices (anti-soliton–soliton and soliton–anti-soliton) defined by

$$t(\lambda) = \mathrm{Tr}_a\, \widetilde{K}^+_a(\lambda)\ \mathcal{T}_a(\lambda)\ , \qquad \bar{t}(\lambda) = \mathrm{Tr}_a\, \widetilde{K}^+_{\bar{a}}(\lambda)\ \overline{\mathcal{T}}_a(\lambda)\ , \tag{2.14}$$

where now

$$\begin{aligned} &T_a(\lambda){=}R_{a\,2L}(\lambda)\bar{R}_{a\,2L-1}(\lambda)\dots R_{a\,2}(\lambda)\bar{R}_{a\,1}(\lambda)\ , \qquad &&\hat{T}_{\bar{a}}(\lambda){=}R_{1\,a}(\lambda)\bar{R}_{2\,a}(\lambda)\dots R_{2L-1\,a}(\lambda)\bar{R}_{2L\,a}(\lambda)\ ,\\ &T_{\bar{a}}(\lambda){=}\bar{R}_{a\,2L}(\lambda)R_{a\,2L-1}(\lambda)\dots \bar{R}_{a2}(\lambda)R_{a1}(\lambda)\ , \qquad &&\hat{T}_a(\lambda){=}\bar{R}_{1a}(\lambda)R_{2a}(\lambda)\dots \bar{R}_{2L-1\,a}(\lambda)R_{2L\,a}(\lambda)\ . \end{aligned} \tag{2.15}$$

Note that, in this case, the number of sites is $2L$ because we want to build an alternating spin chain, which is going to ensure that the Hamiltonian of the model is local. This construction is similar to the one introduced in [38], where however a different notion of $\bar{R}$ was used.
The numerical matrices $\widetilde{K}^-_a$, $\widetilde{K}^-_{\bar{a}}$ are solutions of (1.17), (1.18). The numerical matrices $\widetilde{K}^+_a$ and $\widetilde{K}^+_{\bar{a}}$ are solutions of the following reflection equations:

$$\begin{aligned} &R_{ab}(-\lambda_a+\lambda_b)\ \widetilde{K}^+_a(\lambda_a)^t\ \bar{R}_{ba}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_b(\lambda_b)^t\\ &\qquad = \widetilde{K}^+_b(\lambda_b)^t\ \bar{R}_{ab}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_a(\lambda_a)^t\ R_{ba}(-\lambda_a+\lambda_b) \end{aligned} \tag{2.16}$$

$$\begin{aligned} &R_{ab}(-\lambda_a+\lambda_b)\ \widetilde{K}^+_{\bar{a}}(\lambda_a)^t\ \bar{R}_{ba}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_{\bar{b}}(\lambda_b)^t\\ &\qquad = \widetilde{K}^+_{\bar{b}}(\lambda_b)^t\ \bar{R}_{ab}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_{\bar{a}}(\lambda_a)^t\ R_{ba}(-\lambda_a+\lambda_b) \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\bar{R}_{ab}(-\lambda_a+\lambda_b)\ \widetilde{K}^+_{\bar{a}}(\lambda_a)^t\ R_{ba}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_b(\lambda_b)^t\\ &\qquad = \widetilde{K}^+_b(\lambda_b)^t\ R_{ab}(-\lambda_a-\lambda_b-2i\rho)\ \widetilde{K}^+_{\bar{a}}(\lambda_a)^t\ \bar{R}_{ba}(-\lambda_a+\lambda_b) \end{aligned} \tag{2.18}$$

The commutators

$$[t(\lambda), t(\mu)] = 0 \qquad , \quad [\bar{t}(\lambda), \bar{t}(\mu)] = 0 \qquad \text{and} \quad [t(\lambda), \bar{t}(\mu)] = 0 \tag{2.19}$$

are ensured by the above exchange relations for $\widetilde{K}^+$ and the relations for the monodromy matrices, namely

$$\begin{aligned} R_{ab}(\lambda_a-\lambda_b)\ \mathcal{T}_a(\lambda_a)\ \bar{R}_{ba}(\lambda_a+\lambda_b)\ \mathcal{T}_b(\lambda_b) &= \mathcal{T}_b(\lambda_b)\ \bar{R}_{ab}(\lambda_a+\lambda_b)\ \mathcal{T}_a(\lambda_a)\ R_{ba}(\lambda_a-\lambda_b)\ , &(2.20)\\ R_{ab}(\lambda_a-\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ \bar{R}_{ba}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_b(\lambda_b) &= \overline{\mathcal{T}}_b(\lambda_b)\ \bar{R}_{ab}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ R_{ba}(\lambda_a-\lambda_b)\ , &(2.21)\\ \bar{R}_{ab}(\lambda_a-\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ R_{ba}(\lambda_a+\lambda_b)\ \mathcal{T}_b(\lambda_b) &= \mathcal{T}_b(\lambda_b)\ R_{ab}(\lambda_a+\lambda_b)\ \overline{\mathcal{T}}_a(\lambda_a)\ \bar{R}_{ba}(\lambda_a-\lambda_b)\ . &(2.22) \end{aligned}$$

The relation (2.20) has to be compared with the exchange relation of twisted Yangians based on R-matrix of $\mathcal{Y}(sl_{\mathcal{N}})$. We will come back to this point in section 3.

In the SNP case, one can show [14] that the transfer matrices for $\widetilde{K}^+(u) = \widetilde{K}^-(u) = 1$ exhibit a crossing symmetry, namely

$$t(\lambda) = t(-\lambda - i\rho)\ , \qquad \bar{t}(\lambda) = \bar{t}(-\lambda - i\rho)\ . \tag{2.23}$$

Starting from (2.15) and (2.14), the Hamiltonian of the alternating open spin chain is derived as [14, 17]

$$\mathcal{H} = -\frac{1}{2}\frac{d}{d\lambda} t(\lambda)\ \bar{t}(\lambda)\Big|_{\lambda=0}\ . \tag{2.24}$$

The locality is again ensured by $R(0) = \mathcal{P}$, and integrability is guaranteed by (2.19).

We can explicitly write the open Hamiltonian in terms of the permutation operator and the $R(0)$ and $\bar{R}(0)$ matrices. Let us first introduce some notations:

$$\bar{R}_{ij} = \bar{R}_{ij}(0), \quad \bar{R}'_{ij} = \frac{d}{d\lambda}\bar{R}_{ij}(\lambda)\Big|_{\lambda=0} \quad \text{and} \quad \check{R}'_{ij} = \mathcal{P}_{ij}\,\frac{d}{d\lambda}R_{ij}(\lambda)\Big|_{\lambda=0}. \tag{2.25}$$

After some algebraic manipulations, in particular taking into account that $\mathrm{Tr}_0\,\mathcal{P}_{0i}\;\bar{R}_{0i} \propto \mathbb{I}$, we obtain the following expression for the Hamiltonian (2.24) (for a detailed proof see [14])

$$\begin{aligned}
\mathcal{H} \;\propto\; & \sum_{j=1}^{L} \bar{R}'_{2j-1\;2j}\;\bar{R}_{2j-1\;2j} + \sum_{j=1}^{L-1} \bar{R}_{2j+1\;2j+2}\;\check{R}'_{2j\;2j+2}\;\bar{R}_{2j+1\;2j+2} \\
+ & \sum_{j=1}^{L-1} \bar{R}_{2j+1\;2j+2}\;\bar{R}_{2j-1\;2j}\;\bar{R}'_{2j-1\;2j+2}\;\bar{R}_{2j-1\;2j+2}\;\bar{R}_{2j-1\;2j}\;\bar{R}_{2j+1\;2j+2} \\
+ & \sum_{j=1}^{L-1} \bar{R}_{2j+1\;2j+2}\;\bar{R}_{2j-1\;2j}\;\bar{R}_{2j-1\;2j+2}\;\check{R}'_{2j-1\;2j+1}\;\bar{R}_{2j-1\;2j+2}\;\bar{R}_{2j-1\;2j}\;\bar{R}_{2j+1\;2j+2} \\
+ & \;\mathrm{Tr}_0\,\check{R}'_{0\;2L}\;\bar{R}_{2L-1\;2L}\;\mathcal{P}_{0\;2L-1}\;\bar{R}_{0\;2L-1}\;\bar{R}_{2L-1\;2L} + \bar{R}_{12}\;\check{R}'_{12}\;\bar{R}_{12},
\end{aligned} \tag{2.26}$$

which is indeed local including terms that describe interaction up to four first neighbours.

It is easily shown, acting on (2.15) by full transposition, that $\bar{t}(\lambda) \propto t^{t_1\cdots t_{2L}}(\lambda)$ provided that $(\widetilde{K}^-_a)^t \propto \widetilde{K}^-_{\bar{a}}$. Eigenvectors of $\mathcal{H}$ in (2.26) are determined by sole evaluation of eigenvectors of $t(\lambda)$. We shall therefore only need to consider diagonalisation of $t(\lambda)$ in what follows.

3 Symmetry of the transfer matrix

In the two (SP and SNP) boundary cases, the use of exchange relations for the monodromy matrices allows us to determine the symmetry of the transfer matrix. For simplicity, we fix $K^+(\lambda)$ (or $\widetilde{K}^+(\lambda)$) to be $\mathbb{I}$, leaving $K^-(\lambda)$ (or $\widetilde{K}^-(\lambda)$) free.

3.1 Soliton preserving boundary conditions

In this case, the general form for $K^-(\lambda)$ [35] is conjugated (through a constant matrix) to the following diagonal matrix:

$$K^-(\lambda) = i\xi\mathbb{I} + \lambda\mathbb{E} \qquad \text{with} \quad \mathbb{E} = \mathrm{diag}(\underbrace{+1,...,+1}_{m},\underbrace{-1,...,-1}_{n})\;. \tag{3.1}$$

In the particular case of diagonal solutions, one recovers the scaling limits of the solutions obtained in the "quantum" case in [39]. The monodromy matrix then $\mathcal{T}(\lambda)$ generates a $B(\mathcal{N}, n)$ reflection algebra as studied in [40]. Taking the trace in space a of the relation (2.6), we obtain:

$$(\lambda_a^2 - \lambda_b^2)\,[\,t(\lambda_a)\,,\,\mathcal{T}(\lambda_b)\,] \;=\; (2i\,\lambda_a - \mathcal{N})\,[\,\mathcal{T}(\lambda_a)\,,\,\mathcal{T}(\lambda_b)\,] \tag{3.2}$$

From the asymptotic behaviour ($\lambda \to \infty$) of the R matrix

$$R_{0i}(\lambda) = \lambda\Big(\mathbb{I} + \frac{i}{\lambda}\mathcal{P}_{0i}\Big) , \tag{3.3}$$

we deduce

$$\mathcal{T}(\lambda) = \lambda^{2L+1}\left(\mathbb{E} + \frac{i}{\lambda}\Big(\xi + \sum_{j=1}^{L}\mathcal{B}_{0j}\Big) + O(\frac{1}{\lambda^2})\right) \tag{3.4}$$

where $\mathcal{B}_{0j} = \mathcal{P}_{0j}\,\mathbb{E}_0 + \mathbb{E}_0\,\mathcal{P}_{0j}$. We can write $\mathcal{B}_{0j} = \sum_{\alpha,\beta=1}^{\mathcal{N}} E_{\alpha\beta}\otimes b_j^{\alpha\beta}$, with $E_{\alpha\beta}$ the elementary matrices acting in space 0, and $b_j^{\alpha\beta}$ realizing the $gl(m)\oplus gl(n)$ algebra (in the space j).
Picking up the coefficient of λ_b^{2L} in the relation (3.2), one then concludes that

$$\Big[t(\lambda_a)\,,\,\sum_{j=1}^{L} b_j^{\alpha\beta}\Big] = 0\,, \tag{3.5}$$

i.e. the transfer matrix commutes with the $gl(m)\oplus gl(n)$ algebra.

3.2 Soliton non-preserving boundary conditions

We have already mentioned that the monodromy matrix $\mathcal{T}(\lambda)$ satisfies one of the defining relations (2.20) for the twisted Yangians $\mathcal{Y}^{\pm}(\mathcal{N})$ [36,37]. The general form for $\widetilde{K}^{-}(\lambda)$ is given in proposition 5.3: it is constant and obeys $(\widetilde{K}^{-})^t = \epsilon\widetilde{K}^{-}$ with $\epsilon = \pm 1$.

We need to investigate the asymptotic behaviour ($\lambda \to \infty$) of the R and $\bar{R}$ matrices given by (1.1) and (1.10) respectively:

$$R_{0i}(\lambda) = \lambda\Big(\mathbb{I} + \frac{i}{\lambda}\mathcal{P}_{0i}\Big) \qquad \text{and} \qquad \bar{R}_{0i}(\lambda) = -\lambda\Big(\mathbb{I} + \frac{i}{\lambda}\widehat{\mathcal{P}}_{0i}\Big) , \tag{3.6}$$

where we have introduced

$$\widehat{\mathcal{P}}_{0i} = \rho\,\mathbb{I} - Q_{0i} . \tag{3.7}$$

Accordingly, the monodromy matrices (2.15) take the following form

$$\begin{aligned} T_0(\lambda) &= (-\lambda^2)^L\Big(\mathbb{I} + \frac{i}{\lambda}\sum_{i=1}^{L}\Big(\mathcal{P}_{0,2i} + \widehat{\mathcal{P}}_{0,2i-1}\Big) + O(\frac{1}{\lambda^2})\Big), \\ \hat{T}_{\bar{0}}(\lambda) &= (-\lambda^2)^L\Big(\mathbb{I} + \frac{i}{\lambda}\sum_{i=1}^{L}\Big(\widehat{\mathcal{P}}_{0,2i} + \mathcal{P}_{0,2i-1}\Big) + O(\frac{1}{\lambda^2})\Big) \end{aligned} \tag{3.8}$$

and finally

$$T_0(\lambda)\ \widetilde{K}^{-}\ \hat{T}_{\bar{0}}(\lambda) = \lambda^{4L}\Big(\widetilde{K}^{-} + \frac{i}{\lambda}\sum_{i=1}^{2L}\mathcal{S}_{0i} + O(\frac{1}{\lambda^2})\Big), \tag{3.9}$$

where

$$\mathcal{S}_{0,2i} = \mathcal{P}_{0,2i}\widetilde{K}^{-} + \widetilde{K}^{-}\widehat{\mathcal{P}}_{0,2i} , \qquad \mathcal{S}_{0,2i-1} = \widetilde{K}^{-}\mathcal{P}_{0,2i-1} + \widehat{\mathcal{P}}_{0,2i-1}\widetilde{K}^{-} . \tag{3.10}$$

Similarly to the previous case, one can show from equation (2.20) that (this time for $\widetilde{K}^{-} = 1$ only):

$$\left[t(\lambda_a), \sum_{i=1}^{2L} \mathcal{S}_{0i} \right] = 0 \tag{3.11}$$

In this particular case $\mathcal{S}$ realises the $so(\mathcal{N})$ (or $sp(\mathcal{N})$) generators. When $\mathcal{N}$ is even, as already mentioned, there are two possibilities for the projector Q. More specifically the choice $\theta_0 = 1$ in (1.3) corresponds to the $so(\mathcal{N})$ case, whereas $\theta_0 = -1$ corresponds to the $sp(\mathcal{N})$ case.

Remark: The same construction (open chain with twisted boundary conditions) can be done starting from the $so(\mathcal{N})$, $sp(2\mathcal{N})$ and $osp(\mathcal{M}|2\mathcal{N})$ R-matrix [41]. However, since the fundamental representations of these algebras are self-conjugated, solitons and anti-solitons define the same object. Hence, the "twisted" boundary conditions should be equivalent to open chains with "ordinary" boundary conditions. Indeed, it has been shown in [41] that boundary reflection equations (defining the boundary algebra) and twisted reflection equations (defining the twisted Yangian) are identical.

4 Spectrum of the transfer matrix

Our purpose is to determine the spectrum of the transfer matrix for the $sl(\mathcal{N})$ case.

4.1 Treatment of non-diagonal reflection matrices (SP case)

In the soliton preserving case, the classification of reflection matrices associated to the Yangian $Y(gl(\mathcal{N}))$ has been computed in [35]. It can been recovered from the $sl(\mathcal{M}|\mathcal{N})$ case given in proposition 5.1. Using this classification, it is easy to show the following proposition [42]:

Proposition 4.1 *Let $K(\lambda)$ be any diagonalizable reflection matrix. $D(\lambda)$, the corresponding diagonal reflection matrix, can be written as*

$$D(\lambda) = U^{-1}\, K(\lambda)\, U \tag{4.1}$$

where U, the diagonalization matrix, is constant. Let $t_K(\lambda) = \mathrm{Tr}_a(T_a(\lambda)K_a(\lambda)\hat{T}_a(\lambda))$ and $t_D(\lambda) = \mathrm{Tr}_a(T_a(\lambda)D_a(\lambda)\hat{T}_a(\lambda))$ be the corresponding transfer matrices (we set $K_+ = \mathbb{I}$).
Then, $t_K(\lambda)$ and $t_D(\lambda)$ have the same eigenvalues, their eigenvectors (say v_K and v_D respectively) being related through

$$v_K = U_1 U_2 \dots U_L\, v_D \tag{4.2}$$

Proof: The fact that the diagonalization matrix is a constant (in λ) is a consequence of the classification (see [35] and proposition 5.1). Using the property (1.8), one can show that

$$t_K(\lambda) = U_1 U_2 \dots U_L\, t_D(\lambda)\, (U_1 U_2 \dots U_L)^{-1}\,, \tag{4.3}$$

which is enough to end the proof. ∎

The general treatment (including the super case) for diagonal reflection matrices is done in section

5.4.4. From the above property, this treatment (for diagonal matrices) is enough to obtain the spectrum for *all* the transfer matrices associated to *all* the reflection matrices (provided they are diagonalisable).

As an illustration of this proposition, we compute the eigenvalues associated to the non-diagonal reflection matrix [3, 10]

$$K(\lambda) = \begin{pmatrix} -\lambda + i\xi & 0 & \cdots & 0 & 2k\lambda \\ 0 & c\,\lambda + i\xi & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & c\,\lambda + i\xi & 0 \\ 2k\lambda & 0 & \cdots & 0 & \lambda + i\xi \end{pmatrix} \qquad \text{with} \quad c^2 = 4k^2 + 1 \tag{4.4}$$

It is easy to see that $K(\lambda)$ is diagonalized by the constant matrix

$$U = \begin{pmatrix} -\frac{k}{\xi} & 0 & \cdots & 0 & \frac{k}{\xi} \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & & \ddots & 1 & 0 \\ \frac{c-1}{2c} & 0 & \cdots & 0 & \frac{c+1}{2c} \end{pmatrix}. \tag{4.5}$$

The corresponding diagonal matrix is given by

$$D(\lambda) = c \ \mathrm{diag}(-\lambda + i\xi', \lambda + i\xi', \dots, \lambda + i\xi')\,, \qquad \text{with} \quad \xi' = \frac{\xi}{c}\,, \tag{4.6}$$

in accordance with the classification of reflection matrices. The $t_K(\lambda)$ eigenvalues as well as the Bethe equations are identical to the ones of $t_D(\lambda)$. They can be deduced from the general treatment given in section 5.4.4, taking formally $n_1 = n_2 = \mathcal{N} = 0$ and specifying $m_1 = 1$. They can also be viewed as the scaling limit of quantum groups diagonal solutions [15]. The eigenvectors are related using the formula (4.2), with the explicit form (4.5) for U.

This general procedure can be applied for an arbitrary spin chain, provided the diagonalization matrix is independent from the spectral parameter and commutes, see eq. (1.8), with the R-matrix under consideration. When $K_+(\lambda)$ is not $\mathbb{I}$, this technics can also be used if $K_+(\lambda)$ and $K_-(\lambda)$ can be diagonalized in the same basis [43]. Note however that the classification does not ensure the full generality of such an assumption.

4.2 Pseudo-vacuum and dressing functions

We present below the case when soliton non-preserving boundary conditions are implemented with the simplest choice $\widetilde{K}^{\pm} = \mathbb{I}$. Results for more general choices of $\widetilde{K}^{-}$ can be deduced from the superalgebra case treated in section 5.5.3.

We first derive the pseudo-vacuum eigenvalue denoted as $\Lambda^0(\lambda)$, with the pseudo-vacuum being

$$|\omega_+\rangle = \bigotimes_{i=1}^{2L} |+\rangle_i \qquad \text{where} \qquad |+\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{C}^{\mathcal{N}} . \tag{4.7}$$

Note that the pseudo-vacuum is an exact eigenstate of the transfer matrix (2.3), and $\Lambda^0(\lambda)$ is given by the following expression

$$\Lambda^0(\lambda) = (a(\lambda)\bar{b}(\lambda))^{2L} g_0(\lambda) + (b(\lambda)\bar{b}(\lambda))^{2L} \sum_{l=1}^{\mathcal{N}-2} g_l(\lambda) + (\bar{a}(\lambda)b(\lambda))^{2L} g_{\mathcal{N}-1}(\lambda). \tag{4.8}$$

with

$$a(\lambda) = \lambda + i, \quad b(\lambda) = \lambda, \quad \bar{a}(\lambda) = a(-\lambda - i\rho), \quad \bar{b}(\lambda) = b(-\lambda - i\rho) \tag{4.9}$$

and

$$\begin{aligned} g_l(\lambda) &= \frac{\lambda + \frac{i}{2}(\rho - \theta_0)}{\lambda + \frac{i\rho}{2}}, \qquad 0 \leq l < \frac{\mathcal{N}-1}{2} \\ g_{\frac{\mathcal{N}-1}{2}}(\lambda) &= 1, \qquad\qquad \text{for } \mathcal{N} \text{ odd} \\ g_l(\lambda) &= g_{\mathcal{N}-l-1}(-\lambda - i\rho) \, . \end{aligned} \tag{4.10}$$

We remind that $\rho = \frac{\mathcal{N}}{2}$.
We make at this point the assumption that any eigenvalue of the transfer matrix can be written as

$$\Lambda(\lambda) = (a(\lambda)\bar{b}(\lambda))^{2L} g_0(\lambda) A_0(\lambda) + (b(\lambda)\bar{b}(\lambda))^{2L} \sum_{l=1}^{\mathcal{N}-2} g_l(\lambda) A_l(\lambda) + (\bar{a}(\lambda)b(\lambda))^{2L} g_{\mathcal{N}-1}(\lambda) A_{\mathcal{N}-1}(\lambda) \tag{4.11}$$

where the so-called "dressing functions" $A_i(\lambda)$ need now to be determined.
We immediately get from the crossing symmetry (2.23) of the transfer matrix:

$$A_0(\lambda) = A_{\mathcal{N}-1}(-\lambda - i\rho), \qquad A_l(\lambda) = A_{\mathcal{N}-l-1}(-\lambda - i\rho) \, . \tag{4.12}$$

Moreover, we obtain from the fusion relation (A.10) the following identity, by a comparison of the forms (4.11) for the initial and fused auxiliary spaces:

$$A_0(\lambda + i\rho) A_{\mathcal{N}-1}(\lambda) = 1 \, . \tag{4.13}$$

Gathering the above two equations (4.12), (4.13) we conclude

$$A_0(\lambda) A_0(-\lambda) = 1 \, . \tag{4.14}$$

Finally from equations (B.13) important relations between the dressing functions are entailed for both soliton preserving and soliton non-preserving boundary conditions. In particular,

$$\prod_{l=0}^{\mathcal{N}-1} A_l(\lambda + i(\mathcal{N}-1) - il) = 1. \tag{4.15}$$

Taking into account the constraints (4.12), (4.14) and (4.15) one derives the dressing functions:

$$\begin{aligned} A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda+\lambda_j^{(1)}-\frac{i}{2}}{\lambda+\lambda_j^{(1)}+\frac{i}{2}} \, \frac{\lambda-\lambda_j^{(1)}-\frac{i}{2}}{\lambda-\lambda_j^{(1)}+\frac{i}{2}}, \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda+\lambda_j^{(l)}+\frac{il}{2}} \, \frac{\lambda-\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda-\lambda_j^{(l)}+\frac{il}{2}} \\ &\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}} \, \frac{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}}, \qquad 1 \le l < \frac{\mathcal{N}-1}{2} \end{aligned} \tag{4.16}$$

together with the property $A_l(\lambda) = A_{\mathcal{N}-1-l}(-\lambda - i\rho)$, and, for $\mathcal{N} = 2n+1$:

$$A_n(\lambda) = \prod_{j=1}^{M^{(n)}} \frac{\lambda+\lambda_j^{(n)}+\frac{in}{2}+i}{\lambda+\lambda_j^{(n)}+\frac{in}{2}} \, \frac{\lambda-\lambda_j^{(n)}+\frac{in}{2}+i}{\lambda-\lambda_j^{(n)}+\frac{in}{2}} \, \frac{\lambda+\lambda_j^{(n)}+\frac{in}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(n)}+\frac{in}{2}+\frac{i}{2}} \, \frac{\lambda-\lambda_j^{(n)}+\frac{in}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(n)}+\frac{in}{2}+\frac{i}{2}}, \tag{4.17}$$

Note that the dressing does not depend on the value of θ_0.

The numbers $M^{(l)}$ in (4.17) are related as customary to the eigenvalues of diagonal generators S_l of the underlying symmetry algebra (determined in the previous section), namely

$$S_1 = \frac{1}{2}M^{(0)} - M^{(1)}, \quad S_l = M^{(l-1)} - M^{(l)} \quad \text{with} \quad S_l = \frac{1}{2}(E_{ll} - E_{\overline{ll}}), \quad 1 \le l < \frac{\mathcal{N}-1}{2} \tag{4.18}$$

with $M^{(0)} = 2L$.

Recall that for the $sl(\mathcal{N})$ case the corresponding numbers $M^{(l)}$, see e.g. [11], are given by the following expressions

$$E_{ll} = M^{(l-1)} - M^{(l)}, \qquad l = 1, \dots, \mathcal{N} \tag{4.19}$$

with $M^{(0)} = 2L$ and $M^{(\mathcal{N})} = 0$. If we now impose $M^{(l)} = M^{(\mathcal{N}-l)}$ and consider the differences $E_{ll} - E_{\overline{ll}}$, we end up with relations (4.18), in accordance with the folding of $sl(\mathcal{N})$ leading to $so(\mathcal{N})$ and $sp(\mathcal{N})$ algebras.

4.3 Bethe Ansatz equations

From analyticity requirements one obtains the Bethe Ansatz equations which read as:

4.3.1 sl(2n + 1) algebra

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\; e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\; e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\; e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\; e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
& \quad l=2,\dots,n-1, \\
e_{-\frac{1}{2}}(\lambda_i^{(n)}) &= -\prod_{j=1}^{M^{(n)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n)})\; e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n)}) e_2(\lambda_i^{(n)}-\lambda_j^{(n)})\; e_2(\lambda_i^{(n)}+\lambda_j^{(n)}) \\
& \quad \times \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n-1)})\; e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n-1)}),
\end{aligned}
\tag{4.20}
$$

where we have introduced

$$
e_x(\lambda) = \frac{\lambda + \frac{ix}{2}}{\lambda - \frac{ix}{2}}\,.
$$

It is interesting to note that equations (4.20) are exactly the Bethe Ansatz equations of the $osp(1|\mathcal{N}-1)$ case (see e.g. [14, 19]).

4.3.2 sl(2n) algebra

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\; e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\; e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\; e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\; e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
& \quad l=2,\dots,n-1, \\
e_{-\theta_0}(\lambda_i^{(n)}) &= -\prod_{j=1}^{M^{(n)}} e_2(\lambda_i^{(n)}-\lambda_j^{(n)})\; e_2(\lambda_i^{(n)}+\lambda_j^{(n)}) \\
& \quad \times \prod_{j=1}^{M^{(n-1)}} e_{-1}^2(\lambda_i^{(n)}-\lambda_j^{(n-1)})\; e_{-1}^2(\lambda_i^{(n)}+\lambda_j^{(n-1)}).
\end{aligned}
\tag{4.21}
$$

The Bethe Ansatz equations are essentially the same as the ones obtained from the *folding* of the usual $sl(\mathcal{N})$ Bethe equations (see e.g. [8, 11]) for $M^{(l)} = M^{(\mathcal{N}-l)}$. It can be realised from the study of the underlying symmetry of the model that this *folding* has algebraic origins, as mentioned previously.

5 $sl(\mathcal{M}|\mathcal{N})$ superalgebra

In this section, we generalise the previous approach on (SP and SNP) boundary conditions to the $\mathbb{Z}_2$-graded case based on the $sl(\mathcal{M}|\mathcal{N})$ superalgebra.

5.1 Notations

The $\mathbb{Z}_2$-gradation, depending on a sign $\theta_0 = \pm$, is defined to be $(-1)^{[i]} = \theta_0$ for i an $sl(\mathcal{M})$ index and $(-1)^{[i]} = -\theta_0$ an $sl(\mathcal{N})$ index.
The $sl(\mathcal{M}|\mathcal{N})$ invariant R matrix reads

$$R_{12}(\lambda) = \lambda\mathbb{I} + iP_{12}\ , \tag{5.1}$$

where P is from now on the super-permutation operator (i.e. $X_{21} \equiv PX_{12}P$) such that

$$P = \sum_{i,j=1}^{\mathcal{M}+\mathcal{N}} (-1)^{[j]} E_{ij} \otimes E_{ji} \tag{5.2}$$

The usual super-transposition ^{T} is defined for any matrix $A = \sum_{ij} A^{ij}\ E_{ij}$, by

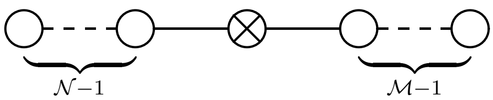
$$A^T = \sum_{ij} (-1)^{[i][j]+[j]}\ A^{ji}\, E_{ij} = \sum_{ij} \left(A^T\right)^{ij}\ E_{ij}\ . \tag{5.3}$$

As for the $sl(\mathcal{N})$ case, we will use a super-transposition ^{t} of the form

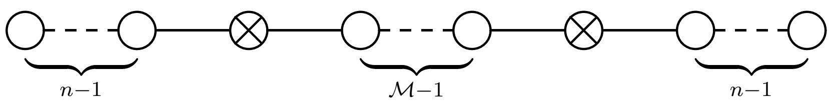
$$A^t = V^{-1}\, A^T\, V\ . \tag{5.4}$$

The convention for θ_0 and the expression of V are chosen accordingly to the selected Dynkin diagram.

Let us recall that for a basic Lie superalgebra, unlike the Lie algebraic case, there exist in general many inequivalent simple root systems (i.e. that are not related by a usual Weyl transformation), and hence many inequivalent Dynkin diagrams. This situation occurs when a simple root system contains at least one isotropic fermionic root. For each basic Lie superalgebra, there is a particular Dynkin diagram which can be considered as canonical: it contains exactly one fermionic root. Such a Dynkin diagram is called distinguished. In the case of $sl(\mathcal{M}|\mathcal{N})$, it has the following form:



In the case of $sl(\mathcal{M}|2n)$ superalgebras, there exists a symmetric Dynkin diagram with two isotropic fermionic simple roots in positions n and $\mathcal{M}+n$:



The generalization of the Weyl group for a basic Lie superalgebra gives a method for constructing all the inequivalent simple root systems and hence all the inequivalent Dynkin diagrams. For more details, see [44–47].

(i) Distinguished Dynkin diagram basis

In this case, we consider that the $sl(\mathcal{M})$ part occupies the 'upper' part of the matrices and corresponds to bosonic degrees of freedom, whereas the $sl(\mathcal{N})$ part occupies the 'lower' part and corresponds to fermionic degrees. More precisely, the gradation takes the form:

$$(-1)^{[i]} = \begin{cases} 1 & \text{for} \quad 1 \leq i \leq \mathcal{M} \\ -1 & \text{for} \quad \mathcal{M}+1 \leq i \leq \mathcal{M}+\mathcal{N}\,, \end{cases} \tag{5.5}$$

and the matrix V reads

$$V = \left(\begin{array}{c|c} V_{\mathcal{M}} & 0 \\ \hline 0 & V_{\mathcal{N}} \end{array} \right) . \tag{5.6}$$

In the above formula, $V_{\mathcal{M}}$ (resp. $V_{\mathcal{N}}$) is the $\mathcal{M} \times \mathcal{M}$ (resp. $\mathcal{N} \times \mathcal{N}$) matrix given in (1.3) for $\theta_0 = +1$.

(ii) Symmetric Dynkin diagram basis

As in the $sl(\mathcal{N})$ case, one has to take $\mathcal{M}$ or $\mathcal{N}$ even. Note that for the odd–odd case no symmetric Dynkin diagram exists and consequently no twisted super-Yangian [48]. Here, we choose $\mathcal{N}$ to be even: $\mathcal{N} = 2n$. The $sl(\mathcal{M})$ part lies in the 'middle' part of the matrices and corresponds to fermionic degrees of freedom, whereas the $sl(\mathcal{N})$ part occupies the 'upper' and 'lower' part and is associated to bosonic degrees of freedom. Correlatively, $\theta_0 = -1$ in this case. The gradation is given by

$$(-1)^{[i]} = \begin{cases} 1 & \text{for} \quad 1 \leq i \leq n \qquad \text{and} \quad \mathcal{M}+n+1 \leq i \leq \mathcal{M}+\mathcal{N} \\ -1 & \text{for} \quad n+1 \leq i \leq \mathcal{M}+n\,, \end{cases} \tag{5.7}$$

while

$$V = \text{antidiag}\Big(\underbrace{1, \dots, 1}_{n+\mathcal{M}}, \underbrace{-1, \dots, -1}_{n} \Big) . \tag{5.8}$$

We will mostly use the distinguished Dynkin diagram basis in the soliton preserving case, and the symmetric one in the soliton non-preserving case. In both cases, the R-matrix obeys the properties stated in section 1.1, with $\bar{R}_{12}(\lambda) = R_{12}^{t_1}(-\lambda - i\rho)$ and $2\rho = \theta_0(\mathcal{M} - \mathcal{N})$. The K-matrices will obey the defining relations stated in section 1.2, and the properties of the transfer matrix (section 2) also hold; the tensor product is now $\mathbb{Z}_2$-graded.

5.2 Classification of reflection matrices for $\mathcal{Y}(\mathcal{M}|\mathcal{N})$

This section is devoted to the classification of reflection matrices for the super-Yangian $\mathcal{Y}(\mathcal{M}|\mathcal{N})$ based on $sl(\mathcal{M}|\mathcal{N})$, both for soliton preserving (prop. 5.1 and 5.2) and for soliton non-preserving boundary conditions (prop. 5.3 and 5.4).

5.2.1 Soliton preserving reflection

Proposition 5.1 *Any bosonic invertible solution of the soliton preserving reflection equation (RE)*

$$R_{12}(\lambda_1-\lambda_2)K_1(\lambda_1)R_{12}(\lambda_1+\lambda_2)K_2(\lambda_2) = K_2(\lambda_2)R_{12}(\lambda_1+\lambda_2)K_1(\lambda_1)R_{12}(\lambda_1-\lambda_2)$$

where $R_{12}(\lambda) = \lambda\,\mathbb{I} + i\,P_{12}$ is the super-Yangian R-matrix, takes the form $K(\lambda) = U\ (i\xi\,\mathbb{I} + \lambda\,\mathbb{E})\,U^{-1}$ where either

(i) $\mathbb{E}$ is diagonal and $\mathbb{E}^2 = \mathbb{I}$ (diagonalisable solutions)

(ii) $\mathbb{E}$ is strictly triangular and $\mathbb{E}^2 = 0$ (non-diagonalisable solutions)

and U is an element of the group $GL(\mathcal{M}) \times GL(\mathcal{N})$. The classification is done up to multiplication by a function of the spectral parameter.

Proof: Firstly it is obvious that for any solution $K(\lambda)$ to the RE, and for any function $f(\lambda)$, the product $f(\lambda)K(\lambda)$ is also a solution to the RE, so that the classification will be done up to multiplication by a function of λ.

Expanding the reflection equation, one rewrites it as:

$$\begin{aligned}[K_2(\lambda_1),\, K_2(\lambda_2)] &= i(\lambda_1+\lambda_2)\,(K_2(\lambda_1)\,K_1(\lambda_2) - K_2(\lambda_2)\,K_1(\lambda_1)) \\ &\quad + i(\lambda_1-\lambda_2)\,(K_1(\lambda_1)\,K_1(\lambda_2) - K_2(\lambda_2)\,K_2(\lambda_1))\end{aligned} \tag{5.9}$$

One then considers the RE with λ_1 and λ_2 exchanged, and sums these two. After multiplication by P_{12}, one gets (for $\lambda_1 \neq \lambda_2$):

$$[K_1(\lambda_1),\, K_1(\lambda_2)] = -[K_2(\lambda_1),\, K_2(\lambda_2)] \tag{5.10}$$

the only solution of which is $[K(\lambda_1),\, K(\lambda_2)] = 0$. In other words, the matrices $K(\lambda)$ at different values of λ's are diagonalisable (or triangularisable) in the same basis and must satisfy

$$(\lambda_1+\lambda_2)\,(K_2(\lambda_1)\,K_1(\lambda_2) - K_2(\lambda_2)\,K_1(\lambda_1)) + (\lambda_1-\lambda_2)\,(K_1(\lambda_1)\,K_1(\lambda_2) - K_2(\lambda_2)\,K_2(\lambda_1)) = 0 \tag{5.11}$$

By setting $\lambda_2 = -\lambda_1$, we get $K(\lambda)K(-\lambda) = k(\lambda)\,\mathbb{I}$ for some function $k(\lambda)$. If one now considers the case of invertible matrices, and since we are looking for solutions up to a multiplicative function, we can take $K(\lambda)K(-\lambda) = \mathbb{I}$, a condition which is generally assumed for reflection matrices.

We first consider the case where these matrices can be diagonalised: $K(\lambda) = U\,D(\lambda)\,U^{-1}$, where U is a group element of $GL(m) \times GL(n)$. Projecting the RE on the basis element $E_{ii} \otimes E_{jj}$, one gets

$$(\lambda_1+\lambda_2)\Big(d_j(\lambda_1)\,d_i(\lambda_2) - d_j(\lambda_2)\,d_i(\lambda_1)\Big) = (\lambda_1-\lambda_2)\Big(d_j(\lambda_2)\,d_j(\lambda_1) - d_i(\lambda_1)\,d_i(\lambda_2)\Big) \tag{5.12}$$

where $D(\lambda) = \mathrm{diag}(d_1(\lambda), d_2(\lambda), \ldots, d_{m+n}(\lambda))$. Since $K(\lambda)$ is supposed invertible, all the d_j's are not zero, and we consider

$$q_{ij}(\lambda) = \frac{d_i(\lambda)}{d_j(\lambda)} \tag{5.13}$$

which obeys

$$(x+y)\Big(q(y)-q(x)\Big)+(x-y)\Big(q(x)q(y)-1\Big)=0\,. \tag{5.14}$$

The solution to this equation is $q(x)=-\frac{x+i\xi}{x-i\xi}$ where ξ is some complex parameter (including $\xi=\infty$), so that, considering $q_{j1}(\lambda)$, we get

$$d_j(\lambda)=-\frac{\lambda+i\xi_j}{\lambda-i\xi_j}\,d_1(\lambda)\ , \qquad \forall\ j \tag{5.15}$$

Requiring $q_{ij}(\lambda)$ to obey the equation (5.14) shows that one must have

$$d_j(\lambda)=\epsilon_j\lambda+i\xi \ \text{ with } \ \epsilon_j=\pm1, \qquad \forall j \tag{5.16}$$

where we have used the invariance under multiplication by a function. This yields the form (i), with $\mathbb{E}=\text{diag}(\epsilon_1,\ldots,\epsilon_{m+n})$.

We now turn to the case $K(\lambda)=U\,T(\lambda)\,U^{-1}$ where $T(\lambda)$ is triangular. The projection of the RE on $E_{ii}\otimes E_{jj}$ shows that the diagonal part of $T(\lambda)$ is still of the form (i). We distinguish two cases: $\mathbb{E}$ has two different eigenvalues (which are ±1), or $\mathbb{E}$ is proportional to $\mathbb{I}$ (and then the diagonal of $T(\lambda)$ is also proportional to $\mathbb{I}$).

If $\mathbb{E}$ has two eigenvalues, we project, in the first auxiliary space, on two diagonal elements E_{jj} and E_{kk} associated to these eigenvalues:

$$(\lambda_1+\lambda_2)\Big((i\xi\pm\lambda_2)T(\lambda_1)-(i\xi\pm\lambda_1)T(\lambda_2)\Big)=(\lambda_1-\lambda_2)\Big(T(\lambda_1)T(\lambda_2)-(i\xi\pm\lambda_1)(i\xi\pm\lambda_2)\Big) \tag{5.17}$$

The difference and the sum of these equations read:

$$\lambda_2\,T(\lambda_1)-\lambda_1\,T(\lambda_2)=\lambda_2-\lambda_1\,, \qquad \lambda_1+\lambda_2\neq0 \tag{5.18}$$

$$i\xi(\lambda_1+\lambda_2)\Big(T(\lambda_1)-T(\lambda_2)\Big)=(\lambda_1-\lambda_2)\Big(T(\lambda_1)T(\lambda_2)+\xi^2-\lambda_1\lambda_2\Big) \tag{5.19}$$

From equation (5.18), one gets

$$\frac{T(\lambda_1)-\mathbb{I}}{\lambda_1}=\frac{T(\lambda_2)-\mathbb{I}}{\lambda_2}=T_0\,, \qquad \textit{i.e.} \qquad T(\lambda)=\mathbb{I}+\lambda\,T_0 \tag{5.20}$$

where T_0 is a triangular matrix. Plugging this solution in equation (5.19), we obtain

$$(i\xi-1)(\lambda_1+\lambda_2)\,T_0=\lambda_1\lambda_2(T_0^2-\mathbb{I})+(\xi^2+1)\,\mathbb{I}, \qquad \forall\lambda_1,\lambda_2 \tag{5.21}$$

whose only (constant) solution is of the form (i) with $i\xi=1$.

We are thus left with the case where the diagonal of $T(\lambda)$ is proportional to the identity matrix: $T(\lambda)=\mathbb{I}+S(\lambda)$ with $S(\lambda)$ strictly triangular. Projecting once more on a diagonal element in the first auxiliary space, we obtain

$$2\Big(\lambda_2\,S(\lambda_1)-\lambda_1\,S(\lambda_2)\Big)=(\lambda_1-\lambda_2)S(\lambda_1)S(\lambda_2) \tag{5.22}$$

$$\Leftrightarrow \quad \frac{S(\lambda_1)}{\lambda_1(2\,\mathbb{I}+S(\lambda_1))}=\frac{S(\lambda_2)}{\lambda_2(2\,\mathbb{I}+S(\lambda_2))}=\sigma \tag{5.23}$$

where σ is strictly triangular. We therefore have $T(\lambda) = \mathbb{I} + 2\lambda\,\sigma(\mathbb{I} - \lambda\sigma)^{-1}$. With this form for $T(\lambda)$, the RE rewrites $(\sigma_1 - \sigma_2)\sigma_1\sigma_2 = 0$, whose solution (for strictly triangular matrices) is given by $\sigma^2 = 0$. Using this property, we get the solution (ii). ∎

Note that the solutions given in this proposition are all of the form $K(\lambda) = i\xi\,\mathbb{I} + \lambda\,\mathcal{E}$ with $\mathcal{E}^2 = \mathbb{I}$ or $\mathcal{E}^2 = 0$ (taking $\mathcal{E} = U\mathbb{E}U^{-1}$).

Proposition 5.2 *Given a solution $K(\lambda) = i\xi\,\mathbb{I} + \lambda\,\mathcal{E}$ to the soliton preserving RE*

$$R_{12}(\lambda_1 - \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2) = K_2(\lambda_2)\ R_{12}(\lambda_1 + \lambda_2)\ K_1(\lambda_1)\ R_{21}(\lambda_1 - \lambda_2)\ , \qquad (5.24)$$

and a solution $\bar{K}(\lambda) = i\xi'\,\mathbb{I} + \lambda\,\mathcal{E}'$ to the anti-soliton preserving RE identical to (5.24), the compatibility condition

$$\bar{R}_{12}(\lambda_1 - \lambda_2)\ K_{\bar{1}}(\lambda_1)\ \bar{R}_{21}(\lambda_1 + \lambda_2)\ K_2(\lambda_2) = K_2(\lambda_2)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ K_{\bar{1}}(\lambda_1)\ \bar{R}_{21}(\lambda_1 - \lambda_2) \qquad (5.25)$$

is solved by $\mathcal{E}' = \mathcal{E}^t$ and $\xi + \xi' = \theta_0 \frac{\mathcal{M}-\mathcal{N}}{2}\,\mathrm{Str}\,\mathcal{E}$.

Proof: Straightforwardly, equation (5.25) is equivalent to

$$\begin{aligned} \mathcal{E}'^t_2 Q_{12} \mathcal{E}_2 &= \mathcal{E}_2 Q_{12} \mathcal{E}'^t_2 &\qquad (5.26)\\ 2(\xi + \xi')[\mathcal{E}_2, Q_{12}] &= [\mathcal{E}_2, Q_{12}\mathcal{E}_2 Q_{12}] &\qquad (5.27) \end{aligned}$$

The first equation yields $\mathcal{E}' = \mathcal{E}^t$. Using $Q_{12}\mathcal{E}_2 Q_{12} = \theta_0 \frac{\mathcal{M}-\mathcal{N}}{2} Q_{12}\ \mathrm{Str}\,\mathcal{E}$ one gets the relation between ξ and ξ'. ∎

5.2.2 Soliton non-preserving reflection

Proposition 5.3 *Any bosonic invertible solution of the soliton non-preserving RE*

$$R_{12}(\lambda_1 - \lambda_2)\ \widetilde{K}_1(\lambda_1)\ \bar{R}_{21}(\lambda_1 + \lambda_2)\ \widetilde{K}_2(\lambda_2) = \widetilde{K}_2(\lambda_2)\ \bar{R}_{12}(\lambda_1 + \lambda_2)\ \widetilde{K}_1(\lambda_1)\ R_{21}(\lambda_1 - \lambda_2) \qquad (5.28)$$

where $R_{12}(\lambda) = \lambda\,\mathbb{I} + i\,P_{12}$ is the super-Yangian R-matrix, is a constant (up to a multiplication by a scalar function) matrix such that $\widetilde{K}^t = \pm\widetilde{K}$.

Proof: Writing the R and $\bar{R}$ matrices in terms of $\mathbb{I}$, P_{12} and Q_{12}, and taking the part of (5.28) which is symmetric in the exchange of λ_1 and λ_2, yields the following equation

$$\widetilde{K}_1(\lambda_1)Q_{12}\widetilde{K}_1(\lambda_2) + \widetilde{K}_1(\lambda_2)Q_{12}\widetilde{K}_1(\lambda_1) = \widetilde{K}_2(\lambda_1)Q_{12}K_2(\lambda_2) + \widetilde{K}_2(\lambda_2)Q_{12}\widetilde{K}_2(\lambda_1) \qquad (5.29)$$

In the same way, exchanging the role of spaces 1 and 2 from the original equation, one gets

$$\widetilde{K}_1(\lambda_1)Q_{12}\widetilde{K}_2(\lambda_2) + \widetilde{K}_2(\lambda_1)Q_{12}\widetilde{K}_1(\lambda_2) = \widetilde{K}_2(\lambda_2)Q_{12}\widetilde{K}_1(\lambda_1) + \widetilde{K}_1(\lambda_2)Q_{12}\widetilde{K}_2(\lambda_1) \qquad (5.30)$$

Transposing both equations (5.29) and (5.30) in space 1 and eliminating P_{12}, one gets after some algebra

$$\widetilde{K}^t(\lambda_2) = f(\lambda_1, \lambda_2)\widetilde{K}(\lambda_1) \qquad \forall\ \lambda_1, \lambda_2 \qquad (5.31)$$

from which the final result follows. ∎

Proposition 5.4 *Given a solution $\widetilde{K}_1$ to the soliton non-preserving RE*

$$R_{12}(\lambda_1-\lambda_2)\ \widetilde{K}_1(\lambda_1)\ \bar{R}_{21}(\lambda_1+\lambda_2)\ \widetilde{K}_2(\lambda_2)=\widetilde{K}_2(\lambda_2)\ \bar{R}_{12}(\lambda_1+\lambda_2)\ \widetilde{K}_1(\lambda_1)\ R_{21}(\lambda_1-\lambda_2)\ , \qquad (5.32)$$

and a solution $\widetilde{K}_{\bar{1}}$ to the CP-conjugate RE identical to (5.32), the compatibility condition

$$\bar{R}_{12}(\lambda_1-\lambda_2)\ \widetilde{K}_{\bar{1}}(\lambda_1)\ R_{21}(\lambda_1+\lambda_2)\ \widetilde{K}_2(\lambda_2)=\widetilde{K}_2(\lambda_2)\ R_{12}(\lambda_1+\lambda_2)\ \widetilde{K}_{\bar{1}}(\lambda_1)\ \bar{R}_{21}(\lambda_1-\lambda_2) \qquad (5.33)$$

is solved by $\widetilde{K}_{\bar{1}}\propto\left(\widetilde{K}_1\right)^{-1}$.

Proof: Straightforward. ■

5.3 Pseudo-vacuum and its dressing

We can determine explicitly the eigenvalue $\Lambda_0(\lambda)$ of the transfer matrix (defined as in section 2) acting on the pseudo-vacuum $|\,\omega_+\rangle$ (which is always bosonic in our conventions). We take here $K^{\pm}=\mathbb{I}$ (resp. $\widetilde{K}^{\pm}=\mathbb{I}$), whilst cases with non trivial $K^{\pm}$ (resp. $\widetilde{K}^{\pm}$) are studied in section 5.4.4 (resp. 5.5.3). $\Lambda_0(\lambda)$ is given by the following expression

$$\Lambda^0(\lambda)=\alpha(\lambda)^L g_0(\lambda)+\beta(\lambda)^L\sum_{l=1}^{\mathcal{M}+\mathcal{N}-2}(-1)^{[l+1]}g_l(\lambda)+\gamma(\lambda)^L(-1)^{[\mathcal{M}+\mathcal{N}]}g_{\mathcal{M}+\mathcal{N}-1}(\lambda) \qquad (5.34)$$

where for (using the notation given in (4.9)):

(i) Soliton preserving boundary conditions with L sites (distinguished Dynkin diagram)

$$\alpha(\lambda)=a^2(\lambda),\quad \beta(\lambda)=\gamma(\lambda)=b^2(\lambda) \qquad (5.35)$$

and

$$\begin{aligned} g_l(\lambda)&=\frac{\lambda(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2})}{(\lambda+\frac{il}{2})(\lambda+\frac{i(l+1)}{2})}\ , \quad l=0,\dots,\mathcal{M}-1\\ g_l(\lambda)&=\frac{\lambda(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2})}{(\lambda+\frac{i(2\mathcal{M}-l-1)}{2})(\lambda+\frac{i(2\mathcal{M}-l)}{2})}\ , \quad l=\mathcal{M},\dots,\mathcal{M}+\mathcal{N}-1 \end{aligned} \qquad (5.36)$$

(ii) Soliton non–preserving boundary conditions with $2L$ sites (symmetric Dynkin diagram)

$$\alpha(\lambda)=\Big(a(\lambda)\bar{b}(\lambda)\Big)^2,\quad \beta(\lambda)=\Big(b(\lambda)\bar{b}(\lambda)\Big)^2,\quad \gamma(\lambda)=\Big(\bar{a}(\lambda)b(\lambda)\Big)^2 \qquad (5.37)$$

and

$$\begin{aligned} g_l(\lambda)&=\frac{\lambda+\frac{i}{2}(\rho+1)}{\lambda+\frac{i\rho}{2}},\qquad 0\le l<\frac{\mathcal{M}+\mathcal{N}-1}{2}\\ g_{\frac{\mathcal{M}+\mathcal{N}-1}{2}}(\lambda)&=1,\qquad \text{if}\quad \mathcal{M}+\mathcal{N}\quad \text{odd}\\ g_l(\lambda)&=g_{\mathcal{N}+\mathcal{M}-l-1}(-\lambda-i\rho). \end{aligned} \qquad (5.38)$$

We remind that $\theta_0=-1$ in that case.

From the exact expression for the pseudo-vacuum eigenvalue, we introduce the following assumption for the structure of the general eigenvalues:

$$\begin{aligned}\Lambda(\lambda) &= \alpha(\lambda)^L g_0(\lambda)A_0(\lambda) + \beta(\lambda)^L \sum_{l=1}^{\mathcal{M}+\mathcal{N}-2} (-1)^{[l+1]} g_l(\lambda)A_l(\lambda) \\ &\quad + \gamma(\lambda)^L (-1)^{[\mathcal{M}+\mathcal{N}-1]} g_{\mathcal{M}+\mathcal{N}-1}(\lambda)A_{\mathcal{M}+\mathcal{N}-1}(\lambda) \end{aligned} \tag{5.39}$$

where the dressing functions $A_i(\lambda)$ need to be determined. The basic constraints that they have to satisfy are the fusion and crossing equations as well as analyticity requirements.

5.4 $sl(\mathcal{M}|\mathcal{N})$ with soliton preserving boundary conditions

From the analyticity of $\Lambda(\lambda)$, one gets

$$\begin{aligned} A_l(-\frac{il}{2}) &= A_{l-1}(-\frac{il}{2}), \qquad l = 1, \dots, \mathcal{M}-1, \\ A_{2\mathcal{M}-l}(-\frac{il}{2}) &= A_{2\mathcal{M}-l-1}(-\frac{il}{2}), \qquad l = \mathcal{M}-\mathcal{N}+1, \dots, \mathcal{M}-1 \end{aligned} \tag{5.40}$$

Gathering the constraints (2.12), (A.10) and (5.40), one can determine the dressing functions, i.e.

$$\begin{aligned} A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda+\lambda_j^{(1)}-\frac{i}{2}}{\lambda+\lambda_j^{(1)}+\frac{i}{2}}\, \frac{\lambda-\lambda_j^{(1)}-\frac{i}{2}}{\lambda-\lambda_j^{(1)}+\frac{i}{2}} \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda+\lambda_j^{(l)}+\frac{il}{2}}\, \frac{\lambda-\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda-\lambda_j^{(l)}+\frac{il}{2}} \\ &\quad \times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}}\, \frac{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}} \qquad l=1,\dots,\mathcal{M}-1 \\ A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+i\mathcal{M}-\frac{il}{2}-i}{\lambda+\lambda_j^{(l)}+i\mathcal{M}-\frac{il}{2}}\, \frac{\lambda-\lambda_j^{(l)}+i\mathcal{M}-\frac{il}{2}-i}{\lambda-\lambda_j^{(l)}+i\mathcal{M}-\frac{il}{2}} \\ &\quad \times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+i\mathcal{M}-\frac{il}{2}+\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+i\mathcal{M}-\frac{il}{2}-\frac{i}{2}}\, \frac{\lambda-\lambda_j^{(l+1)}+i\mathcal{M}-\frac{il}{2}+\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+i\mathcal{M}-\frac{il}{2}-\frac{i}{2}} \\ &\quad l=\mathcal{M},\dots,\mathcal{M}+\mathcal{N}-1 \end{aligned} \tag{5.41}$$

5.4.1 Bethe Ansatz equations for the distinguished Dynkin diagram

From analyticity requirements one obtains the Bethe Ansatz equations,

$$\begin{aligned}
e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\, e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\, e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\
1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\, e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\, e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\
&\quad l=2,\dots,\mathcal{M}-1,\mathcal{M}+1,\dots,\mathcal{M}+\mathcal{N}-2 \\
1 &= \prod_{j=1}^{M^{(\mathcal{M}-1)}} e_{-1}(\lambda_i^{(\mathcal{M})}-\lambda_j^{(\mathcal{M}-1)})\, e_{-1}(\lambda_i^{(\mathcal{M})}+\lambda_j^{(\mathcal{M}-1)}) \\
&\quad\times \prod_{j=1}^{M^{(\mathcal{M}+1)}} e_1(\lambda_i^{(\mathcal{M})}-\lambda_j^{(\mathcal{M}+1)})\, e_1(\lambda_i^{(\mathcal{M})}+\lambda_j^{(\mathcal{M}+1)}) \\
1 &= -\prod_{j=1}^{M^{(\mathcal{M}+\mathcal{N}-2)}} e_{-1}(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}-\lambda_j^{(\mathcal{M}+\mathcal{N}-2)})\, e_{-1}(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}+\lambda_j^{(\mathcal{M}+\mathcal{N}-2)}) \\
&\quad\times \prod_{j=1}^{M^{(\mathcal{M}+\mathcal{N}-1)}} e_2(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}-\lambda_j^{(\mathcal{M}+\mathcal{N}-1)})\, e_2(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}+\lambda_j^{(\mathcal{M}+\mathcal{N}-1)})
\end{aligned} \tag{5.42}$$

We recover here for $\mathcal{M}=2$ and $\mathcal{N}=1$ the Bethe Ansatz equation of the supersymmetric $t-J$ model which corresponds to the $sl(2|1)$ case [49].

5.4.2 Bethe Ansatz equations for arbitrary Dynkin diagrams

We wrote above only the dressing functions that correspond to the distinguished Dynkin diagram. It is however possible to construct the g and dressing functions for all the inequivalent Dynkin diagrams of $sl(\mathcal{M}|\mathcal{N})$.

The inequivalent Dynkin diagrams of the $sl(\mathcal{M}|\mathcal{N})$ superalgebras contain only bosonic root of same square length ("white dots"), usually normalized to 2, and isotropic fermionic roots ("grey dots"). A given diagram is completely characterized by the p-uple of integers $0<n_1<\ldots<n_p<\mathcal{M}+\mathcal{N}$ labelling the positions of the grey dots of the diagram. Formally, we define $n_0=0$ and $n_{p+1}=\mathcal{M}+\mathcal{N}$ although there is actually no root at these positions. Such a diagram defined by the p-uple $(n_i)_{i=1\ldots p}$ corresponds to the superalgebra $sl(\mathcal{M}|\mathcal{N})$ with

$$\mathcal{M}=\sum_{\substack{i \text{ odd}\\ i\le p+1}} n_i-\sum_{\substack{i \text{ even}\\ i<p+1}} n_i \qquad \text{and} \qquad \mathcal{N}=\sum_{\substack{i \text{ even}\\ i\le p+1}} n_i-\sum_{\substack{i \text{ odd}\\ i<p+1}} n_i\,. \tag{5.43}$$

The g functions have a form similar to (5.36), with a change of increasing or decreasing behaviour of the poles each time a grey (fermionic) root is met. Indeed

$$g_l(\lambda)=\frac{\lambda\left(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2}\right)}{\left(\lambda+\frac{i}{2}\delta_l\right)\left(\lambda+\frac{i}{2}(\delta_l+1)\right)}\,,\quad l=0,\dots,\mathcal{M}+\mathcal{N}-1 \tag{5.44}$$

where $\delta_0 = 0$ whilst the δ_l for $l = 1, \dots, \mathcal{M}+\mathcal{N}-1$ are obtained by iteration

$$\delta_l = \begin{cases} \delta_{l-1} & \text{if} \quad l = n_i \quad \text{for some} \quad i \\ \delta_{l-1}+1 & \text{if} \quad n_{2i} < l < n_{2i+1} \quad \text{for some} \quad i \\ \delta_{l-1}-1 & \text{if} \quad n_{2i-1} < l < n_{2i} \quad \text{for some} \quad i \end{cases} \tag{5.45}$$

The Bethe Ansatz equations read, for $\ell = 1, \dots, \mathcal{M}+\mathcal{N}-1$ and $i = 1, \dots, M^{(\ell)}$

$$(1-\langle\alpha_\ell,\alpha_\ell\rangle) \prod_{k=1}^{\mathcal{M}+\mathcal{N}-1} \prod_{j=1}^{M^{(k)}} e_{\langle\alpha_\ell,\alpha_k\rangle}(\lambda_i^{(\ell)}-\lambda_j^{(k)})\, e_{\langle\alpha_\ell,\alpha_k\rangle}(\lambda_i^{(\ell)}+\lambda_j^{(k)}) = \begin{cases} e_1(\lambda_i^{(1)})^{2L} & \ell = 1 \\ 1 & \ell \neq 1 \end{cases} \tag{5.46}$$

where $\langle\alpha_\ell,\alpha_k\rangle$ is the scalar product of the simple roots *associated to the chosen Dynkin diagram.*

5.4.3 Bethe Ansatz equations for the symmetric Dynkin diagram

We give the useful example of the symmetric Dynkin diagram for which $\mathcal{N}$ is even, with the indices ordered as in (5.7). The g functions are in this case

$$\begin{aligned} g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2})}{(\lambda+\frac{il}{2})(\lambda+\frac{i(l+1)}{2})}\,, \quad l = 0,\dots,\mathcal{N}/2-1 \\ g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2})}{(\lambda+\frac{i(\mathcal{N}-l-1)}{2})(\lambda+\frac{i(\mathcal{N}-l)}{2})}\,, \quad l = \mathcal{N}/2,\dots,\mathcal{M}+\mathcal{N}/2-1 \\ g_l(\lambda) &= \frac{\lambda(\lambda+\frac{i(\mathcal{M}-\mathcal{N})}{2})}{(\lambda+\frac{i(l-2\mathcal{M})}{2})(\lambda+\frac{i(l-2\mathcal{M}+1)}{2})}\,, \quad l = \mathcal{M}+\mathcal{N}/2,\dots,\mathcal{M}+\mathcal{N}-1 \end{aligned} \tag{5.47}$$

and it is straightforward to get the A_i's. The Bethe Ansatz equations take the form:

$$\begin{aligned} e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\, e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\, e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,, \\ 1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\, e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\, e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)}) \\ & \quad l = 2,\dots,\mathcal{M}+\mathcal{N}-2,\ \ l \neq \frac{\mathcal{N}}{2},\ \frac{\mathcal{N}}{2}+\mathcal{M} \\ 1 &= \prod_{j=1}^{M^{(l+1)}} e_1(\lambda_i^{(l)}-\lambda_j^{(l+1)})\, e_1(\lambda_i^{(l)}+\lambda_j^{(l+1)}) \prod_{j=1}^{M^{(l-1)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l-1)})\, e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l-1)}) \\ & \quad l = \frac{\mathcal{N}}{2},\ \frac{\mathcal{N}}{2}+\mathcal{M} \\ 1 &= -\prod_{j=1}^{M^{(\mathcal{M}+\mathcal{N}-2)}} e_{-1}(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}-\lambda_j^{(\mathcal{M}+\mathcal{N}-2)})\, e_{-1}(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}+\lambda_j^{(\mathcal{M}+\mathcal{N}-2)}) \\ & \quad \times \prod_{j=1}^{M^{(\mathcal{M}+\mathcal{N}-1)}} e_2(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}-\lambda_j^{(\mathcal{M}+\mathcal{N}-1)})\, e_2(\lambda_i^{(\mathcal{M}+\mathcal{N}-1)}+\lambda_j^{(\mathcal{M}+\mathcal{N}-1)}) \end{aligned} \tag{5.48}$$

The indices of the e's in the products are the entries of the Cartan matrix corresponding to the chosen Dynkin diagram, in accordance with the results obtained in the closed chain case [20, 29, 50].

5.4.4 Non trivial soliton preserving boundary conditions

We come back to the distinguished Dynkin diagram basis and implement non trivial soliton preserving boundary conditions. From the classification given in section 5.2.2, we know that $K^-(\lambda)$ is always conjugated (by a constant matrix U) to a diagonal matrix of the form

$$K(\lambda) = \text{diag}(\underbrace{\alpha, \dots, \alpha}_{m_1}, \underbrace{\beta, \dots, \beta}_{m_2}, \underbrace{\beta, \dots, \beta}_{n_2}, \underbrace{\alpha, \dots, \alpha}_{n_1}) \tag{5.49}$$

As in the section 4.1, it is easy to see that the spectrum and the symmetry of the model depend only on the diagonal (5.49), and not on U. Indeed, when considering two reflection matrices related by a constant conjugation, the corresponding transfer matrices are also conjugated. Thus, property 5.1 ensures that it is enough to consider diagonal $K(\lambda)$ matrices to get the general case. Such a property, which relies on the form of the R-matrix, is a priori valid only in the *rational* $sl(\mathcal{N})$ and $sl(\mathcal{M}|\mathcal{N})$ cases.
For a diagonal solution (5.49) with $m_1 + m_2 = \mathcal{M}$, $n_1 + n_2 = \mathcal{N}$, $\alpha(\lambda) = -\lambda + i\xi$, $\beta(\lambda) = \lambda + i\xi$, and the free boundary parameter ξ, one can compute the new form $\widetilde{g}_l(\lambda)$ of the g-functions entering the expression of $\widetilde{\Lambda}_0(\lambda)$, the new pseudo-vacuum eigenvalue. They take the form:

$$\begin{aligned}
\widetilde{g}_l(\lambda) &= (-\lambda + i\xi)\, g_l(\lambda), \qquad l = 0, \dots, m_1 - 1 \\
\widetilde{g}_l(\lambda) &= (\lambda + i\xi + im_1)\, g_l(\lambda), \qquad l = m_1, \dots, \mathcal{M} + n_2 - 1 \\
\widetilde{g}_l(\lambda) &= (-\lambda + i\xi - im_2 + in_2)\, g_l(\lambda), \qquad l = \mathcal{M} + n_2, \dots, \mathcal{M} + \mathcal{N} - 1
\end{aligned} \tag{5.50}$$

where $g_l(\lambda)$ are given by (5.36). The dressing functions (5.41) keep the same form, but the Bethe Ansatz equations (5.42) are modified (by $K^-(\lambda)$), so that the value of the eigenvalues $\Lambda(\lambda)$ are different from the ones obtained when $K(\lambda) = \mathbb{I}$.

The modifications induced on Bethe Ansatz equations (5.42) are the following:
– The factor $-e^{-1}_{2\xi+m_1}(\lambda)$ appears in the LHS of the ${m_1}^{th}$ Bethe equation.
– The factor $-e^{-1}_{2\xi+m_1-m_2-n_2}(\lambda)$ appears in the LHS of the $(\mathcal{M} + n_2)^{th}$ Bethe equation.

5.5 $sl(\mathcal{M}|\mathcal{N})$ with soliton non-preserving boundary conditions

From equations of the type (B.13) for the supersymmetric case relations between the dressing functions are entailed for both soliton preserving and soliton non-preserving boundary conditions. In particular, for the case that corresponds to the symmetric Dynkin diagram one obtains ($\mathcal{N} = 2n$, while $\mathcal{M}$ can be even or odd),

$$\prod_{l=0}^{n-1} A_l(\lambda - il) \prod_{l=0}^{n-1} A_{\mathcal{M}+n+l}(\lambda + i\mathcal{M} - i(n-1) + il) = \prod_{l=0}^{\mathcal{M}-1} A_{n+l}(\lambda - i(n-1) + il). \tag{5.51}$$

In fact the latter equation is only necessary for the soliton non-preserving boundary conditions.

5.5.1 Dressing functions

As already mentioned we consider here the R matrix, that corresponds to the symmetric Dynkin diagram. Note that the $sl(2n|\mathcal{M})$ case is isomorphic to $sl(\mathcal{M}|2n)$ and entails the same dressing functions and BAE.

From the constraints (4.12), (4.14), (5.51), we conclude that the dressing functions take the form:

$$\begin{aligned}
A_0(\lambda) &= \prod_{j=1}^{M^{(1)}} \frac{\lambda+\lambda_j^{(1)}-\frac{i}{2}}{\lambda+\lambda_j^{(1)}+\frac{i}{2}}\,\frac{\lambda-\lambda_j^{(1)}-\frac{i}{2}}{\lambda-\lambda_j^{(1)}+\frac{i}{2}}, \\
A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda+\lambda_j^{(l)}+\frac{il}{2}}\,\frac{\lambda-\lambda_j^{(l)}+\frac{il}{2}+i}{\lambda-\lambda_j^{(l)}+\frac{il}{2}} \\
&\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}}\,\frac{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}-\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+\frac{il}{2}+\frac{i}{2}}, \qquad l=1,\dots,n-1 \\
A_l(\lambda) &= \prod_{j=1}^{M^{(l)}} \frac{\lambda+\lambda_j^{(l)}+in-\frac{il}{2}-i}{\lambda+\lambda_j^{(l)}+in-\frac{il}{2}}\,\frac{\lambda-\lambda_j^{(l)}+in-\frac{il}{2}-i}{\lambda-\lambda_j^{(l)}+in-\frac{il}{2}} \\
&\times \prod_{j=1}^{M^{(l+1)}} \frac{\lambda+\lambda_j^{(l+1)}+in-\frac{il}{2}+\frac{i}{2}}{\lambda+\lambda_j^{(l+1)}+in-\frac{il}{2}-\frac{i}{2}}\,\frac{\lambda-\lambda_j^{(l+1)}+in-\frac{il}{2}+\frac{i}{2}}{\lambda-\lambda_j^{(l+1)}+in-\frac{il}{2}-\frac{i}{2}}, \qquad n\le l< n+\frac{\mathcal{M}-1}{2}
\end{aligned} \tag{5.52}$$

and $A_l(\lambda)=A_{\mathcal{M}+2n-1-l}(-\lambda-i\rho)$, and for $\mathcal{M}=2m+1$

$$\begin{aligned}
A_k(\lambda) &= \prod_{j=1}^{M^{(k)}} \frac{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}-i}{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}}\,\frac{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}-i}{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}} \\
&\times \frac{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}+\frac{i}{2}}{\lambda+\lambda_j^{(k)}+in-\frac{ik}{2}-\frac{i}{2}}\,\frac{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}+\frac{i}{2}}{\lambda-\lambda_j^{(k)}+in-\frac{ik}{2}-\frac{i}{2}}, \quad k=m+n
\end{aligned} \tag{5.53}$$

Recall that the $sl(2m+1|2n+1)$ case is not examined because there is no symmetric Dynkin diagram and consequently the 'folding' of the algebra can not be implemented. In any case it is known [48] that the twisted super-Yangian does not exist for $sl(2m+1|2n+1)$.

5.5.2 Bethe Ansatz equations

From the analyticity requirements one obtains the Bethe Ansatz equations which read as:

A. $\mathbf{sl(2m+1|2n)}$ superalgebra

$$
\begin{aligned}
e_1(\lambda_i^{(1)})^{2L} &= -\prod_{j=1}^{M^{(1)}} e_2(\lambda_i^{(1)}-\lambda_j^{(1)})\; e_2(\lambda_i^{(1)}+\lambda_j^{(1)}) \prod_{j=1}^{M^{(2)}} e_{-1}(\lambda_i^{(1)}-\lambda_j^{(2)})\; e_{-1}(\lambda_i^{(1)}+\lambda_j^{(2)})\,,\\
1 &= -\prod_{j=1}^{M^{(l)}} e_2(\lambda_i^{(l)}-\lambda_j^{(l)})\; e_2(\lambda_i^{(l)}+\lambda_j^{(l)}) \prod_{\tau=\pm1}\prod_{j=1}^{M^{(l+\tau)}} e_{-1}(\lambda_i^{(l)}-\lambda_j^{(l+\tau)})\; e_{-1}(\lambda_i^{(l)}+\lambda_j^{(l+\tau)})\\
&\quad l=2,\dots,n+m-1,\;\; l\neq n\\
1 &= \prod_{j=1}^{M^{(n+1)}} e_1(\lambda_i^{(n)}-\lambda_j^{(n+1)})\; e_1(\lambda_i^{(n)}+\lambda_j^{(n+1)}) \prod_{j=1}^{M^{(n-1)}} e_{-1}(\lambda_i^{(n)}-\lambda_j^{(n-1)})\; e_{-1}(\lambda_i^{(n)}+\lambda_j^{(n-1)})\\
e_{-\frac{1}{2}}(\lambda_i^{(k)}) &= -\prod_{j=1}^{M^{(k)}} e_2(\lambda_i^{(k)}-\lambda_j^{(k)})\; e_2(\lambda_i^{(k)}+\lambda_j^{(k)})\; e_{-1}(\lambda_i^{(k)}-\lambda_j^{(k)})\; e_{-1}(\lambda_i^{(k)}+\lambda_j^{(k)})\\
&\quad\times\prod_{j=1}^{M^{(k-1)}} e_{-1}(\lambda_i^{(k)}-\lambda_j^{(k-1)})\; e_{-1}(\lambda_i^{(k)}+\lambda_j^{(k-1)})\,,\quad k=m+n
\end{aligned}
\tag{5.54}
$$

Note that the equations (5.54) are the Bethe Ansatz equations of the $osp(2m+1|2n)$ case (see e.g. [19]) apart from the last equation.

B. $\mathbf{sl(2m|2n)}$ superalgebra

The first $n+m-1$ equations are the same as in the previous case see equation (4.20), but the last equation is modified, with again $k=m+n$, to

$$
e_1(\lambda_i^{(k)}) = -\prod_{j=1}^{M^{(k)}} e_2(\lambda_i^{(k)}-\lambda_j^{(k)})\; e_2(\lambda_i^{(k)}+\lambda_j^{(k)}) \prod_{j=1}^{M^{(k-1)}} e_{-1}^2(\lambda_i^{(k)}-\lambda_j^{(k-1)})\; e_{-1}^2(\lambda_i^{(k)}+\lambda_j^{(k-1)})\,. \tag{5.55}
$$

Notice that the Bethe Ansatz equations as in the non supersymmetric case, are essentially the ones obtained from the *folding* of the *symmetric* $sl(\mathcal{M}/\mathcal{N})$ Bethe equations (5.48) for $M^{(l)}=M^{(\mathcal{M}+\mathcal{N}-l)}$. This *folding* has algebraic origins as can be realised from the study of the underlying symmetry of the model (see (4.18), (4.19)). Indeed only half of the $sl(\mathcal{M}|\mathcal{N})$ generators survive after we impose the soliton non-preserving boundary conditions, and these are exactly the generators of the $osp(\mathcal{M}|\mathcal{N})$ algebra.

5.5.3 Non trivial soliton non-preserving boundary conditions

We generalize the above approach to the diagonal case with $\varepsilon=1$ of the classification given in proposition 5.3:

$$
\widetilde{K}^-(\lambda)=\mathrm{diag}(k_1,\dots,k_{\mathcal{M}+\mathcal{N}}) \qquad \text{with} \quad k_{\mathcal{M}+\mathcal{N}+1-j}=\,k_j\,. \tag{5.56}
$$

We will consider only invertible $\widetilde{K}^-$ matrices, so that $k_i\neq 0\ \forall i$.

The g-functions entering the new pseudo-vacuum eigenvalue are modified in the following way:

$$\widetilde{g}_l(\lambda) \;=\; k_{l+1}\, g_l(\lambda), \qquad 0 \le l \le \frac{\mathcal{M}+\mathcal{N}-1}{2} \tag{5.57}$$

where $g_l(\lambda)$ are given by (5.38). The remaining $\widetilde{g}$ are defined by requiring the crossing relation

$$\widetilde{g}_{\mathcal{M}+\mathcal{N}-l}(-\lambda - i\rho) = \widetilde{g}_l(\lambda)\,. \tag{5.58}$$

The dressing functions (5.52) and (5.53) keep the same form, but the LHS of ℓ^{th} Bethe Ansatz equation (given in (5.54) and (5.55)) is multiplied by $k_\ell/k_{\ell+1}$.

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A Survey of fusion

We present here a brief review on fusion for systems with boundaries [51]. In particular we present the fusion procedure for open systems without crossing symmetry developed in [14, 16].

To cover both the SP and SNP cases, we define the action $*$ by:

$$\begin{aligned} &R^*_{ab} = R_{ab}, \quad \bar{R}^*_{ab} = \bar{R}_{ab}, \quad K^*_a = K_a, \quad K^*_{\bar{a}} = K_{\bar{a}}, \quad \text{for SP b.c.} \\ &R^*_{ab} = \bar{R}_{ab}, \quad \bar{R}^*_{ab} = R_{ab}, \quad K^*_a = \widetilde{K}_a, \quad K^*_{\bar{a}} = \widetilde{K}_{\bar{a}}, \quad \text{for SNP b.c.} \end{aligned} \tag{A.1}$$

The K^*_a and $K^*_{\bar{a}}$ matrices are solutions of the reflection (boundary Yang–Baxter) equation [3]

$$R_{ab}(\lambda_a - \lambda_b)\; K^*_a(\lambda_a)\; R^*_{ba}(\lambda_a + \lambda_b)\; K^*_b(\lambda_b) = K^*_b(\lambda_b)\; R^*_{ab}(\lambda_a + \lambda_b)\; K^*_a(\lambda_a)\; R_{ba}(\lambda_a - \lambda_b), \tag{A.2}$$

and they are related by the constraint

$$\bar{R}_{ab}(\lambda_a - \lambda_b)\; K^*_{\bar{a}}(\lambda_a)\; \bar{R}^*_{ba}(\lambda_a + \lambda_b)\; K^*_b(\lambda_b) = K^*_b(\lambda_b)\; \bar{R}^*_{ab}(\lambda_a + \lambda_b)\; K^*_{\bar{a}}(\lambda_a)\; \bar{R}_{ba}(\lambda_a - \lambda_b)\,. \tag{A.3}$$

These equations unify the equations (1.15)–(1.16) with (1.17)–(1.18).

The starting point for both cases (SP and SNP) is the observation that the $\bar{R}$-matrix (1.9) at the special value $\lambda = -i\rho$ yields a one-dimensional projector onto the one-dimensional $sl(\mathcal{N})$-representation present in the decomposition $\mathcal{N} \otimes \bar{\mathcal{N}} = 1 \oplus (\mathcal{N}^2 - 1)$:

$$P^-_{\bar{a}b} = \frac{1}{\mathcal{N}}\, \mathcal{P}^{t_a}_{ab} \;=\; \frac{1}{\mathcal{N}}\, Q_{ab} \;=\; P^-_{a\bar{b}}\,. \tag{A.4}$$

This is related to the fact that the $\bar{R}$ matrix describes the scattering between soliton and anti-soliton. Accordingly, the $(\mathcal{N}^2 - 1)$-dimensional projector is

$$P^+_{\bar{a}b} = 1 - P^-_{\bar{a}b}. \tag{A.5}$$

Note that the soliton-soliton R matrix at $\lambda = -i$ provides a projector onto a $\mathcal{N}$-dimensional space, reflected in $\mathcal{N} \otimes \mathcal{N} = \mathcal{N} \oplus (\mathcal{N}^2 - \mathcal{N})$. Thus, one needs the $\bar{R}$ matrix, even in the SP case, hence the introduction of $\bar{t}(\lambda)$ in both cases.

We will formulate the fusion procedure for both types of boundary conditions we mentioned.

We introduce the fused R-matrices

$$R^*_{<\bar{a}b>1}(\lambda) = P^+_{\bar{a}b}\ \bar{R}^*_{a1}(\lambda)\ R^*_{b1}(\lambda + i\rho)\ P^+_{\bar{a}b}, \qquad R^*_{<b\bar{a}>1}(\lambda) = P^+_{b\bar{a}}\ R^*_{b1}(\lambda)\ \bar{R}^*_{a1}(\lambda + i\rho)\ P^+_{b\bar{a}}, \tag{A.6}$$

and

$$R^*_{1<\bar{a}b>}(\lambda) = P^+_{\bar{a}b}\ R^*_{1b}(\lambda - i\rho)\ \bar{R}^*_{1a}(\lambda)\ P^+_{\bar{a}b}, \qquad R^*_{1<b\bar{a}>}(\lambda) = P^+_{b\bar{a}}\ \bar{R}^*_{1a}(\lambda - i\rho)\ R^*_{1b}(\lambda)\ P^+_{b\bar{a}}. \tag{A.7}$$

They satisfy generalised Yang-Baxter equations with fused indices. Similarly, we use the reflection equation (A.2) and its dual to obtain the fused K matrices

$$\begin{aligned} K^{*-}_{<\bar{a}b>}(\lambda) &= P^+_{\bar{a}b}\ K^{*-}_{\bar{a}}(\lambda)\ R^*_{b\bar{a}}(2\lambda + i\rho)\ K^{*-}_b(\lambda + i\rho)\ P^+_{b\bar{a}}, \\ K^{*+}_{<\bar{a}b>}(\lambda) &= P^+_{b\bar{a}}\ K^{*+}_{\bar{a}}(\lambda)\ R^*_{b\bar{a}}(-2\lambda - 3i\rho)\ K^{*+}_b(\lambda + i\rho)\ P^+_{\bar{a}b}. \end{aligned} \tag{A.8}$$

Both fused matrices obey generalised reflection equations of the type (1.16) and its 'dual' (for more details we refer the reader to [16, 51]). In an analogous way we obtain the $K^*_{<a\bar{b}>}$ matrices by fusing the spaces a and $\bar{b}$. Now that the fused R and K^* matrices are available, we operate the fusion of the transfer matrix (2.3). The fused transfer matrix is defined by

$$t_F(\lambda) = \mathrm{Tr}_{ab}\left\{K^{*+}_{<\bar{a}b>}(\lambda)\ T_{<\bar{a}b>}(\lambda)\ K^{*-}_{<\bar{a}b>}(\lambda)\ \hat{T}^*_{<b\bar{a}>}(\lambda + i\rho)\right\}, \tag{A.9}$$

where the fused T matrices are obtained using (2.4) with fused R matrices (A.6) and (A.7). After some algebra (see e.g. [51]) we end up with:

$$t_F(\lambda) = \zeta^*(2\lambda + 2i\rho)\ \bar{t}(\lambda)\ t(\lambda + i\rho) - \Delta[K^{*+}(\lambda)]\ \delta[T(\lambda)]\ \Delta[K^{*-}(\lambda)]\ \delta[\hat{T}^*(\lambda)], \tag{A.10}$$

where $\zeta^* = \zeta$ and $\bar{\zeta}^* = \bar{\zeta}$ in the SP case, while $\zeta^* = \bar{\zeta}$ and $\bar{\zeta}^* = \zeta$ in the SNP case. Furthermore the 'quantum determinants' are (when we fuse the spaces $\bar{a}$ and b)

$$\begin{aligned} \delta[T(\lambda)] &= \mathrm{Tr}_{ab}\left\{P^-_{\bar{a}b}\ T_{\bar{a}}(\lambda)\ T_b(\lambda + i\rho)\right\} \\ \delta[\hat{T}^*(\lambda)] &= \mathrm{Tr}_{ab}\left\{P^-_{\bar{a}b}\ \hat{T}^*_b(\lambda)\ \hat{T}^*_{\bar{a}}(\lambda + i\rho)\right\} \\ \Delta[K^{*-}(\lambda)] &= \mathrm{Tr}_{ab}\left\{P^-_{b\bar{a}}\ K^{*-}_{\bar{a}}(\lambda)\ R^*_{b\bar{a}}(2\lambda + i\rho)\ K^{*-}_b(\lambda + i\rho)\right\} \\ \Delta[K^{*+}(\lambda)] &= \mathrm{Tr}_{ab}\left\{P^-_{\bar{a}b}\ K^{*+}_b(\lambda + i\rho)\ R^*_{\bar{a}b}(-2\lambda - 3i\rho)\ K^{*+}_{\bar{a}}(\lambda)\right\}. \end{aligned} \tag{A.11}$$

One obtains similar relations when the spaces a and $\bar{b}$ are fused. To compute the quantum determinants explicitly we use the following identities which can be easily proved with the help of unitarity (1.5) and the crossing relation (1.9)

$$P^-_{\bar{a}b}\ R_{\bar{a}m}(\lambda)\ R_{bm}(\lambda + i\rho) = \zeta(\lambda + i\rho)\ P^-_{\bar{a}b}, \tag{A.12}$$

$$P^-_{a\bar{b}}\ R_{am}(\lambda)\ R_{\bar{b}m}(\lambda + i\rho) = \bar{\zeta}(\lambda + i\rho)\ P^-_{a\bar{b}}, \qquad m = 1, \ldots, L^* \tag{A.13}$$

where $L^* = L$ in the SP case and $L^* = 2L$ in the SNP case. One then writes when fusing the spaces $\bar{a}$ and b

$$\delta[T(\lambda)] = \zeta(\lambda + i\rho)^{L^*/2}\zeta^*(\lambda + i\rho)^{L^*/2} , \qquad \delta[\hat{T}^*(\lambda)] = \zeta^*(\lambda + i\rho)^{L^*/2}\zeta(\lambda + i\rho)^{L^*/2} \tag{A.14}$$

whilst, when we fuse the spaces a and $\bar{b}$,

$$\delta[T(\lambda)] = \bar{\zeta}(\lambda + i\rho)^{L^*/2}\bar{\zeta}^*(\lambda + i\rho)^{L^*/2} , \qquad \delta[\hat{T}^*(\lambda)] = \bar{\zeta}^*(\lambda + i\rho)^{L^*/2}\bar{\zeta}(\lambda + i\rho)^{L^*/2} . \tag{A.15}$$

Furthermore, the rôle of ζ and $\bar{\zeta}$ is interchanged in the latter equation depending whether the space V_m belongs to the fundamental representation or to its conjugate. This statement is important if one aims at constructing the alternating spin chain. Finally for the special case $K^- = 1$ and $K^+ = 1$

$$\Delta[K^{*-}(\lambda)] = q^*(2\lambda + i\rho), \qquad \Delta[K^{*+}(\lambda)] = q^*(-2\lambda - 3i\rho) , \tag{A.16}$$

where

$$\begin{aligned} q^*(\lambda) &= q(\lambda) \ \text{ for SP}, \quad q^*(\lambda) = \bar{q}(\lambda) \ \text{ for SNP} \\ q(\lambda) &= \lambda - i\rho, \quad \bar{q}(\lambda) = \lambda + i. \end{aligned} \tag{A.17}$$

B Generalised fusion

We describe a generalised fusion procedure for $sl(\mathcal{N})$ open spin chains [42]. The procedure we use follows the lines of the construction of the Sklyanin determinant for twisted Yangians [37] and reflection algebras [40,52]. The crucial observation here is that for the general case an one dimensional projector can be also obtained by repeating the fusion procedure $\mathcal{N}$ times, this is because $\mathcal{N}^{\otimes\mathcal{N}} = 1 \oplus \dots$. The procedure described in the previous section is basically consequence of the fact that $\mathcal{N} \otimes \bar{\mathcal{N}} = 1 \ \oplus \ (\mathcal{N}^2 - 1)$.

Let us now introduce the following necessary objects for the *generalised* fusion procedure for open spin chains (see also equations (2.13), (2.14) in [53]),

$$T_{<\boldsymbol{a}>} \equiv T_{<a_1 \dots a_{\mathcal{N}}>} = T_{a_1}(\lambda_1) \dots T_{a_{\mathcal{N}}}(\lambda_{\mathcal{N}}), \qquad \hat{T}^*_{<\boldsymbol{a}>} = \hat{T}^*_{a_1}(\lambda_1) \dots \hat{T}^*_{a_{\mathcal{N}}}(\lambda_{\mathcal{N}}) \tag{B.1}$$

where $\lambda_l = \lambda + i(l-1)$, $l = 1, \dots, \mathcal{N}$ and R^* defined in (A.1). For two sets $\{\mu_l\}_{l=1,\dots,\mathcal{N}}$ and $\{\mu'_l\}_{l=1,\dots,\mathcal{N}}$ we also define

$$\mathcal{R}^*_{<\boldsymbol{a}>}(\{\mu_l\}, \{\mu'_l\}) = \overrightarrow{\prod_{k=1,\dots,\mathcal{N}-1}} R^*_{a_{k+1}a_k}(\mu_k - \mu'_{k+1}) \dots R^*_{a_{\mathcal{N}}a_k}(\mu_k - \mu'_{\mathcal{N}}) . \tag{B.2}$$

In particular, $\mathcal{R}_{<\boldsymbol{a}>}(\{\lambda_l\}, \{\lambda_l\})$ is proportional to the antisymmetriser $\mathcal{A}$, i.e. the projector onto a one-dimensional space ($\mathcal{A}^2 = \mathcal{A}$). We also use $\mathcal{A}^+ = I - \mathcal{A}$ and

$$\mathcal{R}^{*+}_{<\boldsymbol{a}>}(\{\mu_l\}) = \overleftarrow{\prod_{k=1,\dots,\mathcal{N}-1}} R^*_{a_k a_{\mathcal{N}}}(-\mu_k - \mu_{\mathcal{N}} - 2i\rho) \dots R^*_{a_k a_{k+1}}(-\mu_k - \mu_{k+1} - 2i\rho) \tag{B.3}$$

By multiplying the four equations

$$
\begin{aligned}
&\mathcal{R}^{*+}_{<\boldsymbol{a}>} \equiv \mathcal{R}^{*+}_{<\boldsymbol{a}>}(\{\lambda_l\}) \; = \; \mathcal{A}\; \mathcal{R}^{*+}_{<\boldsymbol{a}>}(\{\lambda_l\}) + \mathcal{A}^+ \; \mathcal{R}^{*+}_{<\boldsymbol{a}>}(\{\lambda_l\}) \;, \\
&T_{<\boldsymbol{a}>} \; = \mathcal{A}\; T_{<\boldsymbol{a}>} + \mathcal{A}^+ \; T_{<\boldsymbol{a}>} \;, \\
&\mathcal{R}^{*}_{<\boldsymbol{a}>} \equiv \mathcal{R}^{*}_{<\boldsymbol{a}>}(\{\lambda_l\},\{-\lambda_l\}) \; = \; \mathcal{A}\; \mathcal{R}^{*}_{<\boldsymbol{a}>}(\{\lambda_l\},\{-\lambda_l\}) + \mathcal{A}^+ \; \mathcal{R}^{*}_{<\boldsymbol{a}>}(\{\lambda_l\},\{-\lambda_l\}) \;, \\
&\hat{T}^{*}_{<\boldsymbol{a}>} \; = \mathcal{A}\; \hat{T}^{*}_{<\boldsymbol{a}>} + \mathcal{A}^+ \; \hat{T}^{*}_{<\boldsymbol{a}>}
\end{aligned} \tag{B.4}
$$

and keeping in mind that

$$
\mathcal{A}\; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \mathcal{A}^+ = \mathcal{A}\; \mathcal{R}^{*+}_{<\boldsymbol{a}>} \; \mathcal{A}^+ = \mathcal{A}\; T_{<\boldsymbol{a}>} \; \mathcal{A}^+ = \mathcal{A}\; \hat{T}^{*}_{<\boldsymbol{a}>} \; \mathcal{A}^+ = 0 \tag{B.5}
$$

we get

$$
\begin{aligned}
\mathrm{Tr}_{<\boldsymbol{a}>}(\mathcal{R}^{*+}_{<\boldsymbol{a}>} \; T_{<\boldsymbol{a}>} \; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \hat{T}^{*}_{<\boldsymbol{a}>}) \;\; &= \;\; \mathrm{Tr}_{<\boldsymbol{a}>}(\mathcal{A}\; \mathcal{R}^{*+}_{<\boldsymbol{a}>} \; \mathcal{A} \;\; T_{<\boldsymbol{a}>} \; \mathcal{A}\; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \mathcal{A}\; \hat{T}^{*}_{<\boldsymbol{a}>}) \\
&+ \;\; \mathrm{Tr}_{<\boldsymbol{a}>}(\mathcal{A}^+ \; \mathcal{R}^{*+}_{<\boldsymbol{a}>} \; \mathcal{A}^+ \;\; T_{<\boldsymbol{a}>} \; \mathcal{A}^+ \; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \mathcal{A}^* \; \hat{T}^{*}_{<\boldsymbol{a}>}).
\end{aligned} \tag{B.6}
$$

Applying equation

$$
T_a(\lambda_a) \; R^*_{ab}(\lambda_a + \lambda_b) \; \hat{T}^*_b(\lambda_b) = \hat{T}^*_b(\lambda_b) \; R^*_{ab}(\lambda_a + \lambda_b) \; T_a(\lambda_a) \tag{B.7}
$$

recursively we can show that

$$
T_{<\boldsymbol{a}>} \; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \hat{T}^{*}_{<\boldsymbol{a}>} = \mathcal{T}^{*}_{<\boldsymbol{a}>} \tag{B.8}
$$

where

$$
\mathcal{T}^{*}_{<\boldsymbol{a}>} = \overrightarrow{\prod_{k=1,\dots,\mathcal{N}}} \left(\mathcal{T}^*_{a_k}(\lambda_k) \overrightarrow{\prod_{l=k+1,\dots,\mathcal{N}}} R^*_{a_l a_k}(\lambda_l + \lambda_k) \right) \;. \tag{B.9}
$$

However, as discussed in [53] the trace of the above quantity decouples to a product of $\mathcal{N}$ transfer matrices, and therefore the LHS of (B.6) simply becomes $\prod_{l=1}^{\mathcal{N}} t(\lambda_l)$. Taking into account the property,

$$
\mathcal{A}\; \mathcal{O}_{<\boldsymbol{a}>} \; \mathcal{A} = \mathrm{Tr}_{<\boldsymbol{a}>}(\mathcal{A}\; \mathcal{O}_{<\boldsymbol{a}>}) \; \mathcal{A}, \tag{B.10}
$$

we can write the first term of the RHS of (B.6) as product of quantum determinants, which are simply c numbers, i.e.

$$
\mathrm{Tr}_{<\boldsymbol{a}>}(\mathcal{A}\; \mathcal{R}^{*+}_{<\boldsymbol{a}>} \; \mathcal{A} \;\; T_{<\boldsymbol{a}>} \; \mathcal{A}\; \mathcal{R}^{*}_{<\boldsymbol{a}>} \; \mathcal{A}\; \hat{T}^{*}_{<\boldsymbol{a}>}) = \Delta\{K^{*+}(\lambda)\} \; \delta\{T(\lambda)\} \; \Delta\{K^{*-}(\lambda)\} \; \delta\{\hat{T}^*(\lambda)\} \tag{B.11}
$$

where

$$
\begin{aligned}
\Delta\{K^{*+}(\lambda)\} \;\; &= \;\; \mathrm{Tr}_{<\boldsymbol{a}>}\{\mathcal{A}\; \mathcal{R}^{*+}_{<\boldsymbol{a}>}\}, \quad \delta\{T(\lambda)\} = \mathrm{Tr}_{<\boldsymbol{a}>}\{\mathcal{A}\; T_{<\boldsymbol{a}>}\}, \\
\Delta\{K^{*-}(\lambda)\} \;\; &= \;\; \mathrm{Tr}_{<\boldsymbol{a}>}\{\mathcal{A}\; \mathcal{R}^{*}_{<\boldsymbol{a}>}\}, \quad \delta\{\hat{T}^*(\lambda)\} = \mathrm{Tr}_{<\boldsymbol{a}>}\{\mathcal{A}\; \hat{T}^{*}_{<\boldsymbol{a}>}\}.
\end{aligned} \tag{B.12}
$$

Finally the second term of the RHS of (B.6) is simply the fused transfer matrix $\tilde{t}(\lambda)$. Therefore, equation (B.6) can be rewritten as

$$
\tilde{t}(\lambda) = \prod_{l=1}^{\mathcal{N}} t(\lambda_l) - \Delta\{K^{*+}(\lambda)\} \; \delta\{T(\lambda)\} \; \Delta\{K^{*-}(\lambda)\} \; \delta\{\hat{T}^*(\lambda)\}. \tag{B.13}
$$

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